15-853:Algorithms in the Real World

Error Correcting Codes II
- Cyclic Codes
- Reed-Solomon Codes

And now a word from our founder…

Governor Sandford [sic] made a useless rival as you and I saw when in San Francisco, to the State University. I could be no party to such a thing.

- Andrew Carnegie in a letter to Andrew White, Ambassador to Berlin, on the establishment of Stanford University, 1901.

Reed-Solomon: Outline

A (n, k, n-k+1) Reed Solomon Code:

Consider the polynomial

$$p(x) = a_{k-1} x^{k-1} + \cdots + a_1 x + a_0$$

Message: $(a_{k-1}, \ldots, a_1, a_0)$

Codeword: (p(1), p(2), …, p(n))

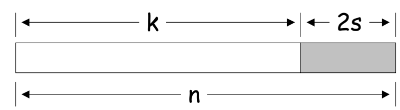
To keep the p(i) fixed size, we use $a_i \in GF(p^r)$

To make the p(i) distinct, $n < p^r$

Any subset of size k of (p(1), p(2), …, p(n)) is enough to reconstruct p(x).

Reed Solomon: Outline

A (n, k, 2s +1) Reed Solomon Code:



Can **detect** 2s errors

Can **correct** s errors

Generally can correct α erasures and β errors if $\alpha + 2\beta \leq 2s$

Reed Solomon: Outline

Correcting s errors:

1. Find k + s symbols that agree on a polynomial p(x). These must exist since originally k + 2s symbols agreed and only s are in error
2. There are no k + s symbols that agree on the wrong polynomial p'(x)
   - Any subset of k symbols will define p'(x)
   - Since at most s out of the k+s symbols are in error, p'(x) = p(x)

Reed Solomon: Outline

Systematic version of Reed-Solomon

$p(x) = a_{k-1} x^{k-1} + \cdots + a_1 x + a_0$

Message: $(a_{k-1}, \ldots, a_1, a_0)$

Codeword: $(a_{k-1}, \ldots, a_1, a_0, p(1), p(2), \ldots, p(2s))$

This has the advantage that if we know there are no errors, it is trivial to decode.

Later we will see that version of RS used in practice uses something slightly different than p(1), p(2), …

This will allow us to use the "**Parity Check**" ideas from linear codes (i.e $Hc^T = 0$?) to quickly test for errors.

RS in the Real World

$(204,188,17)_{256}$: ITU J.83(A)[2]

$(128,122,7)_{256}$: ITU J.83(B)

$(255,223,33)_{256}$: Common in Practice

- Note that they are all byte based (i.e. symbols are from $GF(2^8)$).

Performance on 600MHz Pentium (approx.):

- (255,251) = 45Mbps
- (255,223) = 4Mbps

Dozens of companies sell hardware cores that operate 10x faster (or more)

- (204,188) = 320Mbps (Altera decoder)

Applications of Reed-Solomon Codes

- **Storage**: CDs, DVDs, "hard drives",
- **Wireless**: Cell phones, wireless links
- **Sateline and Space**: TV, Mars rover, …
- **Digital Television**: DVD, MPEG2 layover
- **High Speed Modems**: ADSL, DSL, ..

Good at handling burst errors.

Other codes are better for random errors.

- e.g. Gallager codes, Turbo codes

RS and "burst" errors

Let's compare to Hamming Codes (which are "optimal").

	code bits	check bits
RS **(255, 253, 3)**$_{256}$	2040	16
Hamming **(2^{11}-1, 2^{11}-11-1, 3)**$_2$	2047	11

They can both correct 1 error, but not 2 random errors.
- The Hamming code does this with fewer check bits

However, RS can fix 8 contiguous bit errors in one byte
- Much better than lower bound for 8 arbitrary errors

$$\log\left(1+\binom{n}{1}+\cdots+\binom{n}{8}\right) > 8\log(n-7) \approx 88 \text{ check bits}$$

Galois Fields

The polynomials
$Z_p[x] \bmod p(x)$
where
$p(x) \in Z_p[x]$,
$p(x)$ is irreducible,
and $\deg(p(x)) = n$
form a finite field. Such a field has p^n elements.

These fields are called **Galois Fields** or **GF(p^n).**

The special case n = 1 reduces to the fields Z_p

The multiplicative group of GF(p^n)/{0} is cyclic (this will be important later).

GF(2^n)

Hugely practical!

The coefficients are **bits** {0,1}.

For example, the elements of GF(2^8) can be represented as **a byte**, one bit for each term, and GF(2^{64}) as **a 64-bit word**.
- *e.g.,* $x^6 + x^4 + x + 1$ = 01010011

How do we do addition?

Addition over Z_2 corresponds to xor.
- Just take the xor of the bit-strings (bytes or words in practice). This is dirt cheap

Multiplication over GF(2^n)

If n is small enough can use a table of all combinations.

The size will be $2^n \times 2^n$ (e.g., 64K for GF(2^8)).

Otherwise, use standard shift and add (xor)

Note: dividing through by the irreducible polynomial on an overflow by 1 term is simply a test and an xor.

e.g. 0111 / 1001 = 0111
1011 / 1001 = 1011 xor 1001 = 0010
^ just look at this bit for GF(2^3)

Multiplication over GF(2^5)

```
typedef unsigned char uc;

uc mult(uc a, uc b) {
  int p = a;
  uc r = 0;
  while(b) {
    if (b & 1) r = r ^ p;
    b = b >> 1;
    p = p << 1;
    if (p & 0x10) p = p ^ 0x11B;
  }
  return r;
}
```


Finding inverses over GF(2^n)

Again, if n is small just store in a table.
- Table size is just 2^n.

For larger n, use long division algorithm.
- This is again easy to do with shift and xors.

Galois Field

GF(2^3) with irreducible polynomial: $x^3 + x + 1$

$\alpha = x$ is a generator

α	x	010	**2**
α^2	x^2	100	**3**
α^3	$x + 1$	011	**4**
α^4	$x^2 + x$	110	**5**
α^5	$x^2 + x + 1$	111	**6**
α^6	$x^2 + 1$	101	**7**
α^7	1	001	**1**

Will use this as an example.

Discrete Fourier Transform

Another View of Reed-Solomon Codes

α is a primitive n^{th} root of unity ($\alpha^n = 1$) – a generator

$$T = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \\ 1 & \alpha^2 & \alpha^4 & \cdots & \alpha^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^{n-1} & \alpha^{2(n-1)} & \cdots & \alpha^{(n-1)(n-1)} \end{pmatrix}$$

$$\begin{pmatrix} c_0 \\ \vdots \\ c_{k-1} \\ c_k \\ \vdots \\ c_{n-1} \end{pmatrix} = T \cdot \begin{pmatrix} m_0 \\ \vdots \\ m_{k-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The Discrete Fourier Transform (DFT)

$$m = T^{-1}c$$

Inverse DFT

DFT Example

$\alpha = x$ is 7^{th} root of unity in $GF(2^3)/x^3 + x + 1$
(ie, multiplicative group, which excludes additive inverse)
Recall α = "2", α^2 = "3", … , $\alpha^7 = 1$ = "1"

$$T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 & \alpha^5 & \alpha^6 \\ 1 & \alpha^2 & \alpha^4 & \alpha^6 & & & \\ 1 & \alpha^3 & \alpha^6 & & & & \\ 1 & \alpha^4 & & \ddots & & & \\ 1 & \alpha^5 & & & & & \\ 1 & \alpha^6 & & & & & \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 \\ 1 & 3 & 3^2 & 3^3 & & & \\ 1 & 4 & 4^2 & & & & \\ 1 & 5 & & \ddots & & & \\ 1 & 6 & & & & & \\ 1 & 7 & & & & & 7^6 \end{pmatrix}$$

Should be clear that $c = T \bullet (m_0, m_1, \ldots, m_{k-1}, 0, \ldots)^T$
is the same as evaluating $p(x) = m_0 + m_1x + \ldots + m_{k-1}x^{k-1}$
at n points.

Decoding

Why is it hard?

Brute Force: try k+s choose k + 2s possibilities and solve for each.

Cyclic Codes

A linear code is cyclic if:

$(c_0, c_1, \ldots, c_{n-1}) \in C \Rightarrow (c_{n-1}, c_0, \ldots, c_{n-2}) \in C$

Both **Hamming** and **Reed-Solomon** codes are cyclic.

Note: we might have to reorder the columns to make the code "cyclic".

Motivation: They are more efficient to decode than general codes.

Generator and Parity Check Matrices

Generator Matrix:

A k x n matrix $\boldsymbol{G}$ such that:

$C = \{m \bullet \boldsymbol{G} \mid m \in \Sigma^k\}$

Made from stacking the basis vectors

Parity Check Matrix:

A (n – k) x n matrix **H** such that:

$C = \{v \in \Sigma^n \mid H \bullet v^T = 0\}$

Codewords are the nullspace of H

These **always exist for linear codes**

$H \bullet G^T = 0$

Generator and Parity Check Polynomials

Generator Polynomial:

A degree (n-k) polynomial **g** such that:

$C = \{m \bullet \mathbf{g} \mid m \in \Sigma^k[x]\}$

such that **g** | $x^n - 1$

Parity Check Polynomial:

A degree k polynomial **h** such that:

$C = \{v \in \Sigma^n [x] \mid h \bullet v = 0 \pmod{x^n - 1}\}$

such that **h** | $x^n - 1$

These **always exist for linear cyclic codes**

$h \bullet g = x^n - 1$

Viewing g as a matrix

If $g = g_0 + g_1x + \ldots + g_{n-k}x^{n-k}$

We can put this generator in matrix form:

$$G = \begin{pmatrix} g_0 & g_1 & \cdots & g_{n-k} & 0 & \cdots & 0 \\ 0 & g_0 & \cdots & g_{n-k-1} & g_{n-k} & \cdots & 0 \\ \vdots & & \ddots & & & \ddots & \vdots \\ 0 & 0 & \cdots & g_0 & g_1 & \cdots & g_{n-k} \end{pmatrix}$$

Write $m = m_0 + m_1 x + \ldots m_{k-1}x^{k-1}$ as $(m_0, m_1, \ldots, m_{k-1})$

Then c = mG

g generates cyclic codes

$$G = \begin{pmatrix} g_0 & g_1 & \cdots & g_{n-k} & 0 & \cdots & 0 \\ 0 & g_0 & \cdots & g_{n-k-1} & g_{n-k} & \cdots & 0 \\ \vdots & & \ddots & & & \ddots & \vdots \\ 0 & 0 & \cdots & g_0 & g_1 & \cdots & g_{n-k} \end{pmatrix} = \begin{pmatrix} g \\ xg \\ \vdots \\ x^{k-1}g \end{pmatrix}$$

Codes are linear combinations of the rows.

All but last row is clearly cyclic (based on next row)

Shift of last row is $x^k g \bmod (x^n - 1)$

Consider $h = h_0 + h_1x + \ldots + h_kx^k$ $(gh = x^n - 1)$

$h_0g + (h_1x)g + \ldots + (h_{k-1}x^{k-1})g + (h_kx^k)g = x^n - 1$

$x^kg = -h_k^{-1}(h_0g + h_1(xg) + \ldots + h_{k-1}(x^{k-1}g)) \bmod (x^n - 1)$

This is a linear combination of the rows.

Viewing h as a matrix

If $h = h_0 + h_1x + \ldots + h_kx^k$

we can put this parity check poly. in matrix form:

$$H = \begin{pmatrix} 0 & \cdots & 0 & h_k & \cdots & h_1 & h_0 \\ 0 & \cdots & h_k & h_{k-1} & \cdots & h_0 & 0 \\ \vdots & ⋰ & & & ⋰ & & \vdots \\ h_k & \cdots & h_1 & h_0 & 0 & \cdots & 0 \end{pmatrix}$$

$Hc^T = 0$

Hamming Codes Revisited

The Hamming $(7,4,3)_2$ code.

$g = 1 + x + x^3$ $\qquad h = x^4 + x^2 + x + 1$

$$G = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix} \qquad H = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

$gh = x^7 - 1, \quad GH^T = 0$

The columns are not identical to the previous example Hamming code.

Factors of x^n -1

Intentionally left blank

Another way to write g

Let **α** be a **generator** of $GF(p^r)$.
Let $n = p^r - 1$ (the size of the multiplicative group)
Then we can write a generator polynomial as
$g(x) = (x-\alpha)(x-\alpha^2) \dots (x - \alpha^{n-k})$
Lemma: $g \mid x^n - 1$ ($a \mid b$, means a divides b)
Proof:

- $\alpha^n = 1$ (because of the size of the group)
  $\Rightarrow \alpha^n - 1 = 0$
  $\Rightarrow \alpha$ root of $x^n - 1$
  $\Rightarrow (x - \alpha) \mid x^n - 1$
- similarly for $\alpha^2, \alpha^3, \dots, \alpha^{n-k}$
- therefore $x^n - 1$ is divisible by $(x - \alpha)(x - \alpha^2) \dots$

Back to Reed-Solomon

Consider a generator polynomial $g \in GF(p^r)[x]$, s.t. $g \mid (x^n - 1)$
Recall that $n - k = 2s$ (the degree of g)
Encode:

- $m' = m\, x^{2s}$ (basically shift by 2s)
- $b = m' \pmod g$
- $c = m' - b = (m_{k-1}, \dots, m_0, -b_{2s-1}, \dots, -b_0)$
- Note that c is a **cyclic code** based on g
  - $m' = qg + b$
  - $c = m' - b = qg$

Parity check:

- $h\, c = 0$?

Example

Lets consider the $(7,3,5)_8$ Reed-Solomon code.
We use $GF(2^3)/x^3 + x + 1$

α	x	010	**2**
α^2	x^2	100	**3**
α^3	$x + 1$	011	**4**
α^4	$x^2 + x$	110	**5**
α^5	$x^2 + x + 1$	111	**6**
α^6	$x^2 + 1$	101	**7**
α^7	1	001	**1**

Example RS $(7,3,5)_8$

$g = (x - \alpha)(x - \alpha^2)(x - \alpha^3)(x - \alpha^4)$
$\quad = x^4 + \alpha^3x^3 + x^2 + \alpha x + \alpha^3$
$h = (x - \alpha^5)(x - \alpha^6)(x - \alpha^7)$
$\quad = x^3 + a^3x^3 + a^2x + a^4$
$gh = x^7 - 1$
Consider the message: 110 000 110
$\quad m = (\alpha^4, 0, \alpha^4) = \alpha^4x^2 + \alpha^4$
$\quad m' = x^4m = \alpha^4x^6 + \alpha^4x^4$
$\qquad = (\alpha^4 x^2 + x + \alpha^3)g + (\alpha^3x^3 + \alpha^6x + \alpha^6)$
$\quad c = (\alpha^4, 0, \alpha^4, \alpha^3, 0, \alpha^6, \alpha^6)$
$\qquad = 110\ 000\ 110\ 011\ 000\ 101\ 101$

α	010
α^2	100
α^3	011
α^4	110
α^5	111
α^6	101
α^7	001

$ch = 0 \pmod{x^7 - 1}$

A useful theorem

Theorem: For any β, if $g(\beta) = 0$ then $\beta^{2s}m(\beta) = b(\beta)$
Proof:
$\quad x^{2s}m(x) = g(x)q(x) + b(x)$
$\quad \beta^{2s}m(\beta) = g(\beta)q(\beta) + b(\beta) = b(\beta)$

Corollary: $\beta^{2s}m(\beta) = b(\beta)$ for $\beta \in \{\alpha, \alpha^2, \ldots, \alpha^{2s}\}$
Proof:
$\quad \{\alpha, \alpha^2, \ldots, \alpha^{2s}\}$ are the roots of g by definition.

Fixing errors

Theorem: Any k symbols from c can reconstruct c and hence m
Proof:
We can write 2s equations involving m ($c_{n-1}, \ldots, c_{2s}$) and b ($c_{2s-1}, \ldots, c_0$). These are
$\quad \alpha^{2s}\, m(\alpha) = b(\alpha)$
$\quad \alpha^{4s}\, m(\alpha^2) = b(\alpha^2)$
$\quad \ldots$
$\quad \alpha^{2s(2s)}\, m(\alpha^{2s}) = b(\alpha^{2s})$
We have at most 2s unknowns, so we can solve for them. (I'm skipping showing that the equations are linearly independent).

Efficient Decoding

I don't plan to go into the Reed-Solomon decoding algorithm, other than to mention the steps.