Class Notes for CS 819A4 - Spring 2003

7. Special Classes of Assertions

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An Overview

- Pure
  - independent of the heap
- Intuitionistic
  - never falsified by extending the heap
- Strictly Exact
  - determines unique heap
- Domain-Exact
  - determines unique domain of heap
- Precise
  - For any heap, holds for at most one subset
- Supported
  - For any heap, if it holds for some subset, there is a least subset for which it holds.
- Universally Quantified Intuitionistic
  - $\forall x.\ x \hookrightarrow - \Rightarrow i(x)$.
- Immiscible
  - hold only for disjoint heaps

Pure Assertions

- An assertion is *pure* iff, for any store, it is independent of the heap.
- Syntactically, an assertion is pure if it does not contain **emp**, $\mapsto$, or $\hookrightarrow$.

$$p_1 \wedge p_2 \Rightarrow p_1 \ * \ p_2 \qquad \text{when } p_1 \text{ or } p_2 \text{ is pure}$$

$$p_1 \ * \ p_2 \Rightarrow p_1 \wedge p_2 \qquad \text{when } p_1 \text{ and } p_2 \text{ are pure}$$

$$(p \wedge q) \ * \ r \Leftrightarrow p \wedge (q \ * \ r) \qquad \text{when } p \text{ is pure}$$

$$(p_1 \mathrel{-\!\!*} p_2) \Rightarrow (p_1 \Rightarrow p_2) \qquad \text{when } p_1 \text{ is pure}$$

$$(p_1 \Rightarrow p_2) \Rightarrow (p_1 \mathrel{-\!\!*} p_2) \qquad \text{when } p_1 \text{ and } p_2 \text{ are pure.}$$

Intuitionistic Assertions

- An assertion i is *intuitionistic* iff, for all stores s and heaps h and h':

$$h \subseteq h' \text{ and } s, h \vDash i \text{ implies } s, h' \vDash i.$$

In the following, we assume i and i' are intuitionistic assertions, p is any assertion, and e and e' are expressions:

- Syntactically, the following assertions are intuitionistic:

Any pure assertion	$p * i$
$p \mathrel{-\!\!*} i$	$i \mathrel{-\!\!*} p$
$i \wedge i'$	$i \vee i'$
$\forall v.\ i$	$\exists v.\ i,$

  and as special cases:

$$p * \mathbf{true} \qquad \mathbf{true} \mathrel{-\!\!*} p \qquad e \hookrightarrow e'.$$

- The following laws hold:

$$i * p \Rightarrow i \qquad i \Rightarrow p \mathrel{-\!\!*} i \qquad i * i' \Rightarrow i \wedge i'.$$

  (Note that $i * p \Rightarrow i$ is the weakening law.) Moreover, for arbitrary assertions,

$$p \Rightarrow p * \mathbf{true} \qquad \mathbf{true} \mathrel{-\!\!*} p \Rightarrow p,$$

  so that

$$i * \mathbf{true} \Leftrightarrow i \qquad i \Leftrightarrow \mathbf{true} \mathrel{-\!\!*} i.$$

Intuitionistic Assertions (continued)

- If $p \Rightarrow i$ then

$$p \ * \ \mathbf{true} \Rightarrow i \ * \ \mathbf{true},$$

and since $i \ * \ \mathbf{true} \Leftrightarrow i$,

$$p \ * \ \mathbf{true} \Rightarrow i.$$

Thus, since $p \Rightarrow p \ * \ \mathbf{true}$, $p \ * \ \mathbf{true}$ is the strongest intuitionistic assertion weaker than p.

- If $i \Rightarrow p$ then

$$\mathbf{true} \mathrel{-\!\!*} i \Rightarrow \mathbf{true} \mathrel{-\!\!*} p,$$

and since $i \Leftrightarrow \mathbf{true} \mathrel{-\!\!*} i$,

$$i \Rightarrow \mathbf{true} \mathrel{-\!\!*} p.$$

Thus, since $\mathbf{true} \mathrel{-\!\!*} p \Rightarrow p$, $\mathbf{true} \mathrel{-\!\!*} p$ is the weakest intuitionistic assertion stronger than p.

$$
\begin{array}{l}
\langle \text{int assert} \rangle ::= \langle \text{pure assert} \rangle \\
\quad \mid \overset{\text{i}}{\neg} \langle \text{int assert} \rangle \\
\quad \mid \langle \text{int assert} \rangle \wedge \langle \text{int assert} \rangle \mid \langle \text{int assert} \rangle \vee \langle \text{int assert} \rangle \\
\quad \mid \langle \text{int assert} \rangle \overset{\text{i}}{\Rightarrow} \langle \text{int assert} \rangle \mid \langle \text{int assert} \rangle \overset{\text{i}}{\Leftrightarrow} \langle \text{int assert} \rangle \\
\quad \mid \forall \langle \text{var} \rangle . \langle \text{int assert} \rangle \mid \exists \langle \text{var} \rangle . \langle \text{int assert} \rangle \\
\quad \mid \langle \text{exp} \rangle \overset{\text{i}}{\mapsto} \langle \text{exp} \rangle \\
\quad \mid \langle \text{int assert} \rangle \ * \ \langle \text{int assert} \rangle \\
\quad \mid \langle \text{int assert} \rangle \mathrel{-\!\!*} \langle \text{int assert} \rangle
\end{array}
$$

where

$$e \overset{\text{i}}{\mapsto} e' \overset{\text{def}}{=} e \hookrightarrow e' \overset{\text{def}}{=} (e \mapsto e') \ * \ \mathbf{true}$$

$$\overset{\text{i}}{\neg} p \overset{\text{def}}{=} \mathbf{true} \mathrel{-\!\!*} (\neg p)$$

$$p \overset{\text{i}}{\Rightarrow} q \overset{\text{def}}{=} \mathbf{true} \mathrel{-\!\!*} (p \Rightarrow q)$$

$$p \overset{\text{i}}{\Leftrightarrow} q \overset{\text{def}}{=} \mathbf{true} \mathrel{-\!\!*} (p \Leftrightarrow q).$$

Strictly Exact Assertions (Yang)

- An assertion q is strictly exact iff, for all s, h, and h', $s, h \vDash q$ and $s, h' \vDash q$ implies $h = h'$.
- Syntactically, assertions built from expressions using **emp**, $\mapsto$, and $*$ are strictly exact.

Then:

$$((q * \mathbf{true}) \wedge p) \Rightarrow (q * (q \mathrel{-\!\!*} p)) \quad \text{when } q \text{ is strictly exact.}$$

For example,

$$(((e \mapsto e') * \mathbf{true}) \wedge p) \Rightarrow ((e \mapsto e') * ((e \mapsto e') \mathrel{-\!\!*} p)).$$

This is the fourth axiom schema for $\mapsto$, which is used to derive the backward-reasoning rule for lookup (LKBR) from the large form (LKLG).

Domain-Exact Assertions

- An assertion q is domain-exact iff, for all s, h, and h', $s, h \vDash q$ and $s, h' \vDash q$ implies $\operatorname{dom} h = \operatorname{dom} h'$.
- Strictly exact assertions are domain-exact.
- Syntactically, assertions built from expressions using **emp**, $\mapsto$, $-$, and $*$ are domain-exact.

Precise Assertions

- An assertion q is *precise* iff, for all s and h, there is at most one $h' \subseteq h$ such that $s, h' \vDash q$ is true.
- Domain-exact assertions are precise.
- Assertions such as

  $$\exists\alpha.\ \mathsf{list}\ \alpha\ (\mathsf{i}, \mathbf{nil}) \qquad\qquad \exists\tau.\ \mathsf{tree}\ \tau\ (\mathsf{i})$$

  are precise.
- If p and q are precise, then $p\ *\ q$ is precise.
- If q is precise and $p \Rightarrow q$ is valid, then p is precise.
- Any assertion that is both intuitionistic and precise is unsatisfiable.
- When q is precise, the semidistributive laws given earlier become full distributive laws:

  $$(p_1\ *\ q) \wedge (p_2\ *\ q) \Rightarrow (p_1 \wedge p_2)\ *\ q$$

  when q is precise

  $$\forall x.\ (p\ *\ q) \Rightarrow (\forall x.\ p)\ *\ q$$

  when x not free in q, and q is precise.

Proof (of the first distributive law): Suppose $s, h \vDash (p_1\ *\ q) \wedge (p_2\ *\ q)$. Then there is an $h_1 \subseteq h$ such that $s, h - h_1 \vDash p_1$ and $s, h_1 \vDash q$, and there is an $h_2 \subseteq h$ such that $s, h - h_2 \vDash p_2$ and $s, h_2 \vDash q$. Then, since q is precise, $h_1 = h_2$, and $h - h_1 = h - h_2$, so that $s, h - h_1 \vDash p_1 \wedge p_2$, and $s, h \vDash (p_1 \wedge p_2)\ *\ q$.

Supported Assertions

- An assertion q is *supported* iff, for all s, h_1, and h_2,
  - if $h_1 \cup h_2$ is a function, and $s, h_1 \vDash q$ and $s, h_2 \vDash q$ are true,
  - then there is an h' such that $h' \subseteq h_1$, $h' \subseteq h_2$, and $s, h' \vDash q$ is true.
- An assertion q is *supported* iff, for all s and h, if the set

$$H = \{\, h' \mid h' \subseteq h \text{ and } s, h' \vDash q \,\}$$

  is nonempty, then it has a least element.

Proof (that the first definition implies the second): Suppose that q satisfies the first definition, fix s and h, and let h_1 be a member of of H with minimum domain size, and h_2 be any member of H. Since h_1 and h_2 are both subsets of h, $h_1 \cup h_2$ must be a function. Then the first definition guarantees that there is an $h' \in H$ that is a subset of both h_1 and h_2. But h' must be equal to h_1, since otherwise it would have a smaller domain size. Thus $h_1 \subseteq h_2$ for every $h_2 \in H$.

Supported Assertions (continued)

- If q is precise, then q is supported.
- q is supported iff $q * \mathbf{true}$ is supported.

Proof: Suppose q is supported, $h_1 \cup h_2$ is a function, $s, h_1 \vDash q * \mathbf{true}$ and $s, h_2 \vDash q * \mathbf{true}$. Then there are $h_1' \subseteq h_1$ and $h_2' \subseteq h_2$ such that $s, h_1' \vDash q$ and $s, h_2' \vDash q$, and since q is supported, there is an h' that is a subset of h_1' and h_2', and therefore h_1 and h_2, such that $s, h' \vDash q$, and therefore $s, h' \vDash q * \mathbf{true}$.

Suppose $q * \mathbf{true}$ is supported, $h_1 \cup h_2$ is a function, $s, h_1 \vDash q$ and $s, h_2 \vDash q$. Then $s, h_1 \vDash q * \mathbf{true}$ and $s, h_2 \vDash q * \mathbf{true}$, and since $q * \mathbf{true}$ is supported, there is a common subset h' of h_1 and h_2 such that $s, h' \vDash q * \mathbf{true}$. But then there is a subset h'' of h', and therefore of h_1 and h_2, such that $s, h'' \vDash q$.

The analogous conjecture, that q is supported iff $\mathbf{true} \mathrel{-\!\!*} q$ is supported, fails in both directions. Suppose q is $1 \hookrightarrow 1 \vee 2 \hookrightarrow 2 \vee \mathbf{emp}$. Then q is supported, since the empty heap is the least heap in which it holds, but $\mathbf{true} \mathrel{-\!\!*} q$ is not supported, since it holds for the heaps $[\,1\colon 1\,]$ and $[\,2\colon 2\,]$, but not for their only common subheap, which is the empty heap.

On the other hand, suppose q is $1 \hookrightarrow 1 \vee 2 \mapsto 2$. Then $\mathbf{true} \mathrel{-\!\!*} q$ is supported, since $[\,1\colon 1\,]$ is the least heap for which it holds, but q is not supported, since it holds for $[\,1\colon 1\,]$ and $[\,2\colon 2\,]$, but not for the empty heap.

Supported Assertions (continued)

- When q is supported and the remaining assertions are intuitionistic, the semidistributive laws given earlier become full distributive laws:

$$(p_1 \ * \ q) \wedge (p_2 \ * \ q) \Rightarrow (p_1 \wedge p_2) \ * \ q$$

when p_1 and p_2 are intuitionistic and q is supported

$$\forall x.\ (p \ * \ q) \Rightarrow (\forall x.\ p) \ * \ q$$

when p is intuitionistic, q is supported,
and x is not free in q.

Proof (of the first distributive law): Suppose $s, h \vDash (p_1 \ * \ q) \wedge (p_2 \ * \ q)$. Then there is an $h_1 \subseteq h$ such that $s, h - h_1 \vDash p_1$ and $s, h_1 \vDash q$, and there is an $h_2 \subseteq h$ such that $s, h - h_2 \vDash p_2$ and $s, h_2 \vDash q$. Then, since q is supported, and $h_1 \cup h_2$ is a subset of h (and therefore a function), there is a common subset h' of h_1 and h_2, and therefore of h, such that $s, h' \vDash q$. Then $h - h_1$ and $h - h_2$ are both subsets of $h - h'$, and since p_1 and p_2 are intuitionistic, $s, h - h' \vDash p_1$ and $s, h - h' \vDash p_2$. Therefore $s, h - h' \vDash (p_1 \wedge p_2)$, and $s, h \vDash (p_1 \wedge p_2) \ * \ q$.

The Precising Operation (Yang)

$$\Pr\, p \stackrel{\text{def}}{=} p \wedge \neg(p \ * \ \neg\, \mathbf{emp}).$$

- If p is supported, then $\Pr\, p$ is precise.

Proof: Suppose p is supported, and $h_1, h_2 \subseteq h$ are such that $s, h_1 \vDash \Pr\, p$ and $s, h_2 \vDash \Pr\, p$.

We must show $h_1 = h_2$. Assume the contrary. Since $s, h_1 \vDash p$, $s, h_2 \vDash p$, and p is supported, there is a common subset h' of h_1 and h_2 such that $s, h' \vDash p$. Since $h_1 \neq h_2$, however, h' must be a proper subset of h_i for $i = 1$ or $i = 2$. Thus $h_i = h' \cdot (h_i - h')$, where $s, h_i - h' \vDash \neg\, \mathbf{emp}$. Then $s, h_i \vDash p \ * \ \neg\, \mathbf{emp}$, which contradicts $s, h_i \vDash \Pr\, p$.

- If p is precise, then $\Pr\, p \Leftrightarrow p$.

Proof: Obviously, $\Pr\, p \Rightarrow p$. To show the opposite implication when p is precise, assume $s, h \vDash p$. If $s, h \vDash p \ * \ \neg\, \mathbf{emp}$ held, then there would be a proper subset h' of h such that $s, h' \vDash p$, which would contradict the preciseness of p. Thus $s, h \vDash \neg(p \ * \ \neg\, \mathbf{emp})$.

The conjecture that, when p is supported, $\Pr\, p$ is the weakest precise assertion stronger than p, is false. Suppose p is the supported assertion $1 \mapsto 1 \vee \mathbf{emp}$. Then $1 \mapsto 1$ is a precise assertion stronger than p, but $\Pr\, p$, which is equivalent to $\mathbf{emp}$, is not weaker than $1 \mapsto 1$.

Precisification (continued)

- $\Pr\,(p * \mathbf{true}) \Rightarrow p$.

Proof: Suppose $s, h \vDash \Pr\,(p * \mathbf{true})$. Then $s, h \vDash p * \mathbf{true}$ and $s, h \vDash \neg(p * \mathbf{true} * \neg\,\mathbf{emp})$, and there is an $h' \subseteq h$ such that $s, h' \vDash p$. If $h' = h$, we are done; otherwise, h' is a proper subset of h, so that $s, h \vDash p * \mathbf{true} * \neg\,\mathbf{emp}$, which contradicts $s, h \vDash \neg(p * \mathbf{true} * \neg\,\mathbf{emp})$.

- $p \Rightarrow \Pr\,(p * \mathbf{true})$ when p is precise.

Proof: Suppose $s, h \vDash p$. Then $s, h \vDash p * \mathbf{true}$. Moreover, $s, h \vDash \neg(p * \mathbf{true} * \neg\,\mathbf{emp})$, for otherwise $s, h \vDash p * \mathbf{true} * \neg\,\mathbf{emp}$ would imply that there is a proper subset h' of h such that $s, h' \vDash p$, which would contradict the preciseness of p.

- $(\Pr\,q) * \mathbf{true} \Rightarrow q$ when q is intuitionistic.

Proof: Suppose $s, h \vDash (\Pr\,q) * \mathbf{true}$. Then there is an $h' \subseteq h$ such that $s, h' \vDash \Pr\,q$. Then $s, h' \vDash q$, and since q is intuitionistic, $s, h \vDash q$.

- $q \Rightarrow (\Pr\,q) * \mathbf{true}$ when q is supported.

Proof: Suppose $s, h \vDash q$. Since q is supported, there is a least $h' \subseteq h$ such that $s, h' \vDash q$. Then $s, h' \vDash \Pr\,q$, since otherwise $s, h' \vDash q * \neg\,\mathbf{emp}$, which would imply that a proper subset h'' of h' would satisfy $s, h'' \vDash q$, contradicting the leastness of h'. Thus $s, h \vDash (\Pr\,q) * \mathbf{true}$.

Universally Quantified Intuitionistic Assertions

Suppose i is intuitionistic, and let p be

$$\forall x.\ \mathsf{allocated}(x) \Rightarrow i,$$

where

$$\mathsf{allocated}(x) \equiv x \hookrightarrow -,$$

or, more generally,

$$s, h \vDash \mathsf{allocated}(x) \text{ iff } s\,x \in \mathrm{dom}\,h.$$

Then

$$\mathbf{emp} \Rightarrow p$$
$$(p\ *\ p) \Rightarrow p.$$

Immiscible Assertions

- Assertions p and q are *immiscible* iff, for all stores s and heaps h and h' such that $h \cup h'$ is a function, if $s, h \models p$ and $s, h' \models q$ then $h \perp h'$.
- If i and j are intuitionistic and $\neg(i \wedge j)$ is valid, then

  $$\forall x.\ \mathsf{allocated}(x) \Rightarrow i \qquad \text{and} \qquad \forall x.\ \mathsf{allocated}(x) \Rightarrow j$$

  are immiscible.
- If p' and q' are immiscible and $p \Rightarrow p'$ and $q \Rightarrow q'$ are valid, then p and q are immiscible.

If p and q are immiscible, then

$$(p\ *\ \mathbf{true}) \wedge (q\ *\ r) \Rightarrow q\ *\ ((p\ *\ \mathbf{true}) \wedge r)$$

is valid.

Relevance Conjunction (Bornat)

We define the logical operator $\bullet$ by:

$$s, h \vDash p_0 \bullet p_1 \text{ iff } \exists h_0, h_1.\ h_0 \cup h_1 = h \text{ and } s, h_0 \vDash p_0 \text{ and } s, h_1 \vDash p_1.$$

Note that h_0 and h_1 need not have disjoint domains.

Laws for Relevance Conjunction

$$p \Rightarrow p \bullet p \quad \text{(contraction)}$$

$$p * q \Rightarrow p \bullet q$$

$$p \bullet q \Rightarrow p * q \quad \text{when } p \text{ and } q \text{ are immiscible}$$

$$p \wedge q \Rightarrow p \bullet q$$

$$i \bullet j \Rightarrow i \wedge j \quad \text{when } i \text{ and } j \text{ are intuitionistic}$$

$$p_1 \bullet p_2 \Leftrightarrow p_2 \bullet p_1$$

$$(p_1 \bullet p_2) \bullet p_3 \Leftrightarrow p_1 \bullet (p_2 \bullet p_3)$$

$$p \bullet \mathbf{emp} \Leftrightarrow p$$

$$(p_1 \vee p_2) \bullet q \Leftrightarrow (p_1 \bullet q) \vee (p_2 \bullet q)$$

$$(p_1 \wedge p_2) \bullet q \Rightarrow (p_1 \bullet q) \wedge (p_2 \bullet q)$$

$$(\exists x.\ p_1) \bullet p_2 \Leftrightarrow \exists x.\ (p_1 \bullet p_2) \quad \text{when } x \text{ not free in } p_2$$

$$(\forall x.\ p_1) \bullet p_2 \Rightarrow \forall x.\ (p_1 \bullet p_2) \quad \text{when } x \text{ not free in } p_2$$

$$(p_1 \bullet p_2) * q \Rightarrow (p_1 * q) \bullet (p_2 * q)$$

$$(p_1 * \mathbf{true}) \bullet (p_2 * \mathbf{true}) \Rightarrow (p_1 \bullet p_2) * \mathbf{true}$$

$$\frac{p_1 \Rightarrow p_2 \qquad q_1 \Rightarrow q_2}{p_1 \bullet q_1 \Rightarrow p_2 \bullet q_2} \quad \text{(monotonicity)}$$

Also, if p and q are precise, then $p \bullet q$ is precise.

Precise Dags

For $\tau \in$ S-exps, we define

$$\mathsf{pdag}\, \tau\, (i)$$

by:

$$\mathsf{pdag}\, a\, (i) \text{ iff } \mathbf{emp} \wedge i = a$$

$$\mathsf{pdag}\, (\tau_1 \cdot \tau_2)\, (i) \text{ iff}$$
$$\exists i_1, i_2.\ \ i \mapsto i_1, i_2 \ * \ (\mathsf{pdag}\, \tau_1\, (i_1) \bullet \mathsf{pdag}\, \tau_2\, (i_2))$$

It can be proved by induction on τ that $\mathsf{pdag}\, \tau\, (i)$ is precise.

Moreover,

$$\mathsf{pdag}\, \tau\, (i) \ * \ \mathbf{true} \Leftrightarrow \mathsf{dag}\, \tau\, (i).$$

This proof is also by induction on τ. In the base case

$$\begin{aligned} \mathsf{pdag}\, a\, (i) \ * \ \mathbf{true} &\Leftrightarrow (\mathbf{emp} \wedge i = a) \ * \ \mathbf{true} \\ &\Leftrightarrow i = a \\ &\Leftrightarrow \mathsf{dag}\, a\, (i). \end{aligned}$$

Precise Dags (continued)

The induction step is

$$
\begin{aligned}
&\mathsf{pdag}\,(\tau_1 \cdot \tau_2)\,(i) \;*\; \mathbf{true} \\
&\quad \Leftrightarrow (\exists i_1, i_2.\;\; i \mapsto i_1, i_2 \;*\; (\mathsf{pdag}\,\tau_1\,(i_1) \;\bullet\; \mathsf{pdag}\,\tau_2\,(i_2))) \;*\; \mathbf{true} \\
&\quad \Leftrightarrow \exists i_1, i_2.\;\; i \mapsto i_1, i_2 \;*\; (\mathsf{pdag}\,\tau_1\,(i_1) \;\bullet\; \mathsf{pdag}\,\tau_2\,(i_2)) \;*\; \mathbf{true} \\
&\quad \Leftrightarrow \exists i_1, i_2.\;\; i \mapsto i_1, i_2 \;* \\
&\qquad ((\mathsf{pdag}\,\tau_1\,(i_1) \;*\; \mathbf{true}) \;\bullet\; (\mathsf{pdag}\,\tau_2\,(i_2) \;*\; \mathbf{true})) \\
&\quad \Leftrightarrow \exists i_1, i_2.\;\; i \mapsto i_1, i_2 \;*\; (\mathsf{dag}\,\tau_1\,(i_1) \;\bullet\; \mathsf{dag}\,\tau_2\,(i_2)) \\
&\quad \Leftrightarrow \exists i_1, i_2.\;\; i \mapsto i_1, i_2 \;*\; (\mathsf{dag}\,\tau_1\,(i_1) \wedge \mathsf{dag}\,\tau_2\,(i_2)) \\
&\quad \Leftrightarrow \mathsf{dag}\,\tau\,(i).
\end{aligned}
$$

From this result, we can also obtain

$$\mathsf{pdag}\,\tau\,(i) \Leftrightarrow \Pr\,(\mathsf{dag}\,\tau\,(i)).$$

Moreover, since $\mathsf{pdag}\,\tau(\mathsf{i}) \Rightarrow \mathsf{dag}\,\tau(\mathsf{i})$, we have the following specification of copytree:

$$
\begin{aligned}
&\{\mathsf{p} \wedge \mathsf{pdag}\,\tau(\mathsf{i})\} \\
&\{\mathsf{p} \wedge \mathsf{dag}\,\tau(\mathsf{i})\} \\
&\mathsf{copytree}(\mathsf{i};\mathsf{j}) \\
&\{\mathsf{p} \;*\; \mathsf{tree}\,\tau(\mathsf{j})\}.
\end{aligned}
$$