Non-holonomic Constraints and Lie Brackets

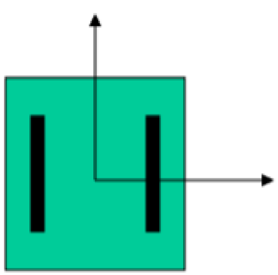
Howie Choset, Hannah Lyness

Underactuated Mechanisms

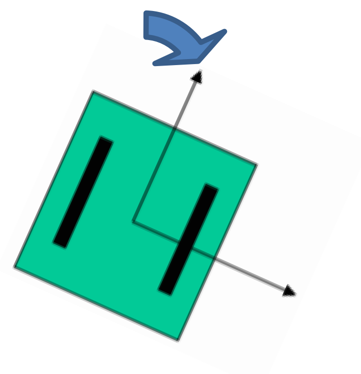
- Redundancy = a couple of extra
- Hyper-redundancy = many extra
- Underactuated = too few?

Definition

- Non-holonomic constraint: a non-integrable constraint
- Example: A constraint on velocity does not induce a constraint on position.
- For a wheeled robot, it can instantaneously move in some directions (forward and backwards), but not others (side to side).

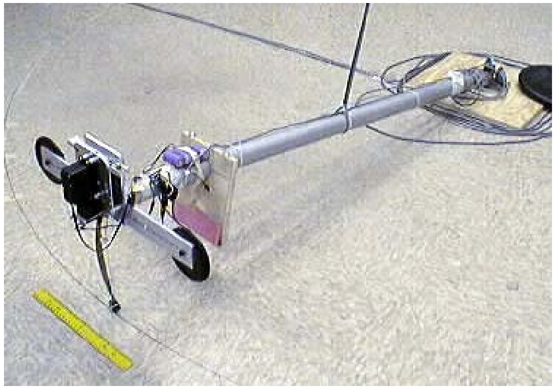


The robot can instantly move forward and back, but it can not move to the left or right without the wheels slipping.

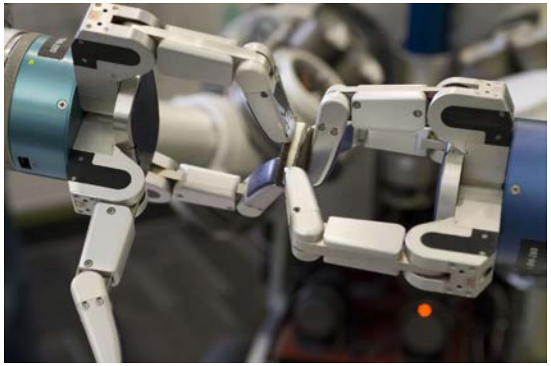


To go to the right, the robot must first turn, and then drive forward.

Other examples of systems with non-holonomic constraints



Hopping robots.
(CMU's BowLeg Hopper)



Manipulation with a robotic hand
(CMU's HERB)

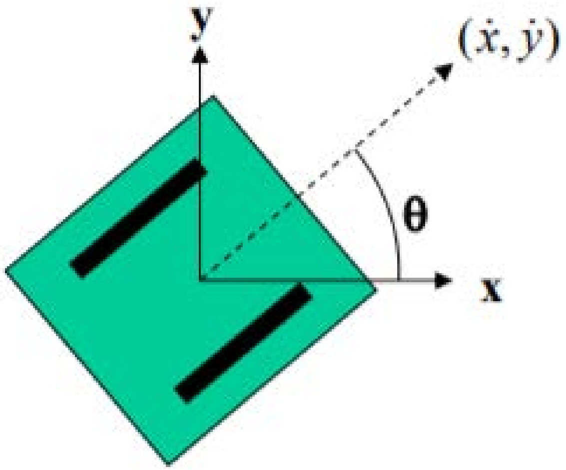
Untethered space robots (conservation of angular momentum is the constraint)
(NASA's AERcam)

What about holonomic systems?

- A person walking is an example of a holonomic system. You can instantly step to the right or left, as well as going forwards or backwards. In other words, your velocity in the plane is not restricted.
- An omni-wheel is a holonomic system-it can roll forwards and sideways.

How do we represent the constraint mathematically?

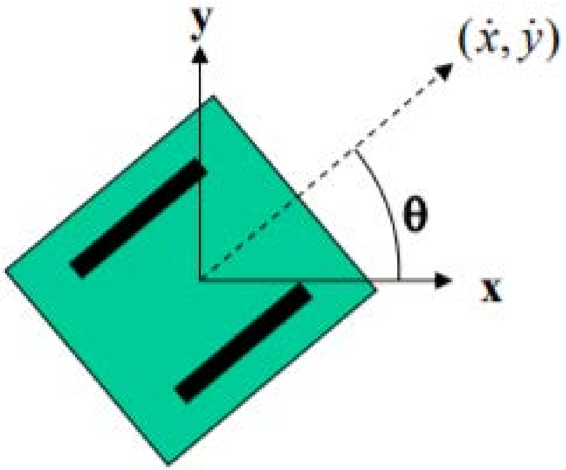
- We write a constraint equation.
  - For a differential drive, this is:
    $\dot{y}cos\theta - \dot{x}sin\theta = 0$
  - What does this equation tell us?
    The direction we can't move in
    - So if theta is 0, the velocity in y=0
    - If theta is 90, the velocity in x=0



How do we represent the constraint mathematically?

- We can also write the relationships in matrix form with q the position vector and q dot as the velocity vector.

$$q = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \qquad \dot{q} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix}$$



- Now our constraint (w) can be expressed as: $w(q) = [-sin\theta \; cos\theta \; 0]$

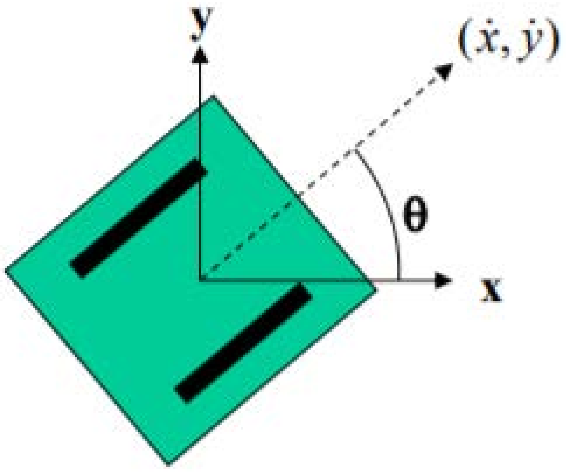
How do we represent the constraint mathematically?

These are equivalent:

$$\dot{y}cos\theta - \dot{x}sin\theta = 0$$

and

$$w(q)\dot{q} = [-sin\theta\ cos\theta\ 0]\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{bmatrix} = 0$$



Lie Brackets

- Lie Bracket: $[g_1, g_2] = \frac{\partial g_2}{\partial q} g_1 - \frac{\partial g_1}{\partial q} g_2$
- A Lie Bracket takes two n dimensional vectors and returns a new n-vector.
- We can use Lie Brackets to reveal additional motions of our system.
- Recall a partial derivative:
  - Let g $= \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $q = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$
  - Then the partial derivative of g with respect to q is:

$$\frac{\partial g}{\partial q} = \begin{bmatrix} \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y} & \frac{\partial a}{\partial \theta} \\ \frac{\partial b}{\partial x} & \frac{\partial b}{\partial y} & \frac{\partial b}{\partial \theta} \\ \frac{\partial c}{\partial x} & \frac{\partial c}{\partial y} & \frac{\partial c}{\partial \theta} \end{bmatrix}$$

Lie Bracket – 2-Wheeled Cart Example

- The states of our system are x, y and θ. We express this as:
  - $q = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}.$
- The constraint on our system (that we found before) is:
  - $w_1 = [-\sin(\theta)\ \cos(\theta)\ 0]$
- The allowable motions of our system are:
  - $g_1 = \begin{bmatrix} cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix}$ (motion in the direction of theta)
  - $g_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (rotation in place)

Lie Bracket – 2-Wheeled Cart Example

- The allowable motions of our system are:
  - $g_1 = \begin{bmatrix} cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix}$ (motion in the direction of theta)
  - $g_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (rotation in place)
- Lie Bracket: $[g_1, g_2] = \frac{\partial g_2}{\partial q} g_1 - \frac{\partial g_1}{\partial q} g_2$

$$[g_1, g_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -\sin(\theta) \\ 0 & 0 & \cos(\theta) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin(\theta) \\ -\cos(\theta) \\ 0 \end{bmatrix}$$

- This new motion is perpendicular to our wheels. We can still not have instantaneous velocity in this direction, but through a combination of movements, this result demonstrates that we can position perpendicular to our wheels.

Method for analyzing non-holonomic motion

- Determine your states (q vector)
- Determine your constraints (w vectors) or ways that you *can't* move
- Convert the constraints into locally allowable motions (g vectors)
    - Must be perpendicular to constraints
- Apply Lie Bracket to g vectors to determine all possible motions.
    - If, after you apply the Lie Bracket, you find that you have n linearly independent columns, then you can control your robot in all n variables.

Lie Bracket – 4-Wheeled Simplified Ackerman Example

- The states of our system are x, y, θ and φ.
  - $q = \begin{bmatrix} x \\ y \\ \theta \\ \varphi \end{bmatrix}$.
- The constraints on our system are:
  - $w_1 = [-\sin(\theta) \quad \cos(\theta) \quad 0 \quad 0]$
    (motion perpendicular to the back wheels)
  - $w_2 = [-\sin(\theta+\varphi) \quad \cos(\theta+\varphi) \quad lcos(\varphi) \quad 0]$
    (motion perpendicular to the front wheels)
- w_2 comes from examining the movement of the front wheels with respect to the reference point of the car:

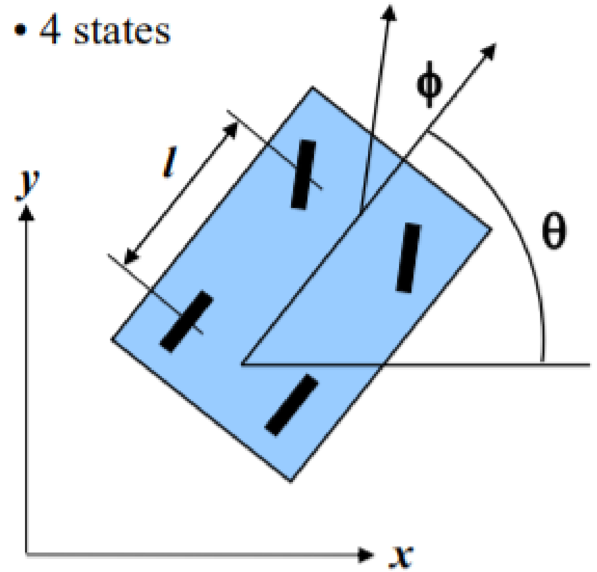
$$-\dot{x}_f \sin(\theta+\varphi) + \dot{y}_f \cos(\theta+\varphi) = 0$$
$$x_f = x + lcos\theta$$
$$y_f = y + lsin\theta$$
$$\dot{x}_f = \dot{x} - lsin\theta\dot{\theta}$$
$$\dot{y}_f = \dot{y} + lcos\theta\dot{\theta}$$
$$-(\dot{x} - lsin\theta\dot{\theta})\sin(\theta+\varphi) + (\dot{y} + lcos\theta\dot{\theta})\cos(\theta+\varphi) = 0$$
$$-\sin(\theta+\varphi)\dot{x} + \cos(\theta+\varphi)\,\dot{y} + lcos\varphi\dot{\theta} = 0$$

Comes from identity
$\cos(A-B) = cosAcosB + sinAsinB$

•2 constraints (front and rear wheels)

•2 inputs (steering and gas pedal)

• 4 states



Lie Bracket – 4-Wheeled Simplified Ackerman Example

- The states of our system are x, y, θ and φ.
  - $q = \begin{bmatrix} x \\ y \\ \theta \\ \varphi \end{bmatrix}.$
- The constraints on our system are:
  - $w_1 = [-\sin(\theta) \quad \cos(\theta) \quad 0 \quad 0]$
    (motion perpendicular to the back wheels)
  - $w_2 = [-\sin(\theta+\varphi) \quad \cos(\theta+\varphi) \quad lcos(\varphi) \quad 0]$
    (motion perpendicular to the front wheels)
- The allowable motions of our system are:
  - $g_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ (rotation of steering wheel)
  - $g_2 = \begin{bmatrix} cos(\theta) \\ \sin(\theta) \\ \frac{1}{l}\tan(\varphi) \\ 0 \end{bmatrix}$ (motion for a fixed phi)

g_2 comes from determining what motion is perpendicular to both w_1 and w_2:

$$[-\sin(\theta+\varphi) \quad \cos(\theta+\varphi) \quad lcos(\varphi) \quad 0]\begin{bmatrix} cos(\theta) \\ \sin(\theta) \\ ? \\ 0 \end{bmatrix} = 0$$

$$-\sin(\theta+\varphi)cos\theta + \cos(\theta+\varphi)\, sin\theta + lcos(\varphi) * ? = 0$$

$$-sin\varphi + lcos(\varphi) * ? = 0$$

$$? = \frac{1}{l}\tan\varphi$$

Comes from identity

$\sin(A-B) = sinAcosB - cosAsinB$

Lie Bracket – 4-Wheeled Simplified Ackerman Example

- We want four linearly independent g vectors because we have 4 states. We have 2 so we need 2 more.
- $g_3 = [g_1, g_2] = \frac{\partial g_2}{\partial q} g_1 - \frac{\partial g_1}{\partial q} g_2$
  - Note that $\frac{\partial g_1}{\partial q} = 0$ since g_1 is a constant
  - Additionally, we only need the last column of $\frac{\partial g_2}{\partial q}$ since the first three rows of g_1 are 0.

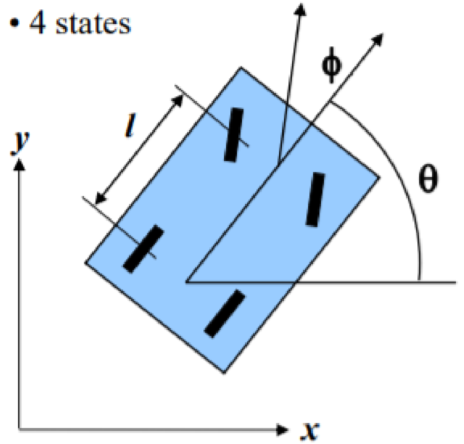
$$g_3 = \begin{bmatrix} X & X & X & 0 \\ X & X & X & 0 \\ X & X & X & \frac{1}{l\cos^2(\varphi)} \\ X & X & X & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ \frac{1}{l}\tan(\varphi) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{l\cos^2(\varphi)} \\ 0 \end{bmatrix}$$

- We found a vector that is linearly independent of our other allowable motions. This new vector only has values in the theta state, meaning we can get to any arbitrary theta. Still not instantaneously, but through a series of maneuvers.

•2 constraints (front and rear wheels)

•2 inputs (steering and gas pedal)

• 4 states



* Here X corresponds to values we don't bother to calculate because we will not need them

Lie Bracket – 4-Wheeled Simplified Ackerman Example

- We want four linearly independent g vectors because we have 4 states. We have 3 now so we need 1 more.
- $g_4 = [g_2, g_3] = \frac{\partial g_3}{\partial q} g_2 - \frac{\partial g_2}{\partial q} g_3$
  - Again, we capitalize on knowing where products will be 0 to speed the process along

$$g_4 = \begin{bmatrix} 0 & 0 & 0 & X \\ 0 & 0 & 0 & X \\ 0 & 0 & 0 & X \\ 0 & 0 & 0 & X \end{bmatrix} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ \frac{1}{l}\tan(\varphi) \\ 0 \end{bmatrix} - \begin{bmatrix} X & X & -\sin(\theta) & X \\ X & X & cos(\theta) & X \\ X & X & 0 & X \\ X & X & 0 & X \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \frac{1}{lcos^2(\varphi)} \\ 0 \end{bmatrix}$$

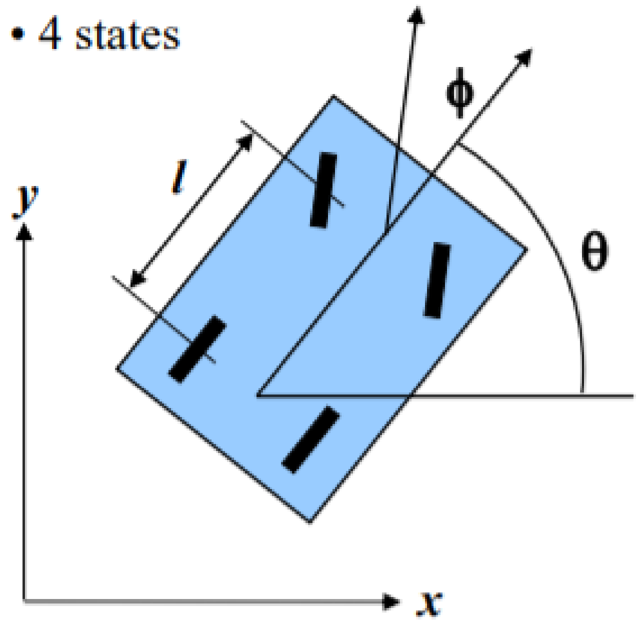
$$= \begin{bmatrix} \frac{\sin(\theta)}{lcos^2(\varphi)} \\ \frac{-\cos(\theta)}{lcos^2(\varphi)} \\ 0 \\ 0 \end{bmatrix}$$

- We found a vector that is linearly independent of our other allowable motions. This new vector only has values in the x and y states. This means that we can position anywhere, though not instantaneously.

•2 constraints (front and rear wheels)

•2 inputs (steering and gas pedal)

• 4 states



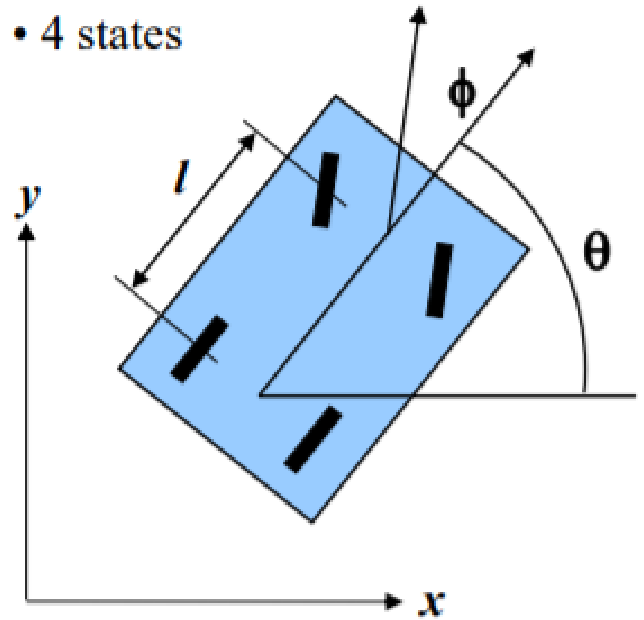
* Here X corresponds to values we don't bother to calculate because we will not need them

Lie Bracket – 4-Wheeled Simplified Ackerman Example

- We found four linearly independent g vectors, which means that our robot is able to position, orient and steer without limits in our space thought a series of incremental movements.

$$[g_1, g_2, g_3, g_4] = \begin{bmatrix} \cos(\theta) & 0 & 0 & \dfrac{\sin(\theta)}{l\cos^2(\varphi)} \\ \sin(\theta) & 0 & 0 & \dfrac{-\cos(\theta)}{l\cos^2(\varphi)} \\ \dfrac{1}{l}\tan(\varphi) & 0 & \dfrac{1}{l\cos^2(\varphi)} & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

- 2 constraints (front and rear wheels)
- 2 inputs (steering and gas pedal)
- 4 states



* Here X corresponds to values we don't bother to calculate because we will not need them