15-864 Advanced Computer Graphics, Doug James (CMU)

Radiosity Equations: Petrov–Galerkin Discretization (PGD)

Petrov-Galerkin discretizations are the most common approach for solving integral equations, and it can be applied to the Radiosity equation

$$B(x') = E(x') + \frac{1}{\pi}\rho(x')\int_S B(x)\,\underbrace{V(x,x')\,G(x,x')}_{\equiv VG \text{ throughout.}}\,dA_x$$

as follows.

Discrete Approximation: The first step is to choose a TRIAL SPACE to represent the solution/unknown function. Let this space be spanned by the basis functions $\{\phi_j(x)\}_{j=1}^N$ so that we have the approximations

$$\circledast \quad \begin{cases} B(x) = \sum_{k=1}^N B_k \phi_k(x) \\ E(x) = \sum_k E_k \phi_k(x) \\ \rho(x) = \sum_k \rho_k \phi_k(x) \end{cases} \leftarrow \text{(in general, } \rho(x) \text{ may be approximated by another space of functions.)}$$

Later we will assume that $\phi_k(x)$ are piecewise constant basis functions, but lets consider the general case for now.

Substituting $\circledast$ into the radiosity eqn & rearranging we have

$$0 = \sum_{k=1}^N \phi_k(x')\left\{ B_k - E_k - \frac{1}{\pi}\rho_k \sum_{j=1}^N B_j \left(\int_S \phi_j(x)\, VG\, dA_x\right)\right\} \equiv R(x')$$

Weighted Residual Method: Petrov-Galerkin discretion is essentially a weighted residual method, where "residual" refers to a quantity that should (tend to) be zero in an accurate approximation. Here the residual equation is $0 = R(x')$, $x' \in S$. Given that $R(x')=0$, then so will any multiple or integral of R. PGD chooses a second TRIAL SPACE of basis functions to integrate against the residual. Let the trial space be span $\{\psi_i\}_{i=1}^N$ such that PGD results in:

$$0 = \int_S R(x')\,\psi_i(x')\,dA_{x'}, \quad i = 1..N.$$

Common PGD Approaches: Choosing the test space : $\text{span}(\{\psi_i\}_{i=1}^N)$.

Galerkin approximation is the most common approach used in radiosity, and corresponds to taking the test space the same as the trial space:

(TEST) $\psi_i(x) = \phi_i(x)$ (TRIAL)

This has the benefit of lead to symmetric form factors from an energy balance perspective (as we will see).

Collocation approximation: is a common approach outside radiosity, & corresponds to satisfying $R(x_i^*)=0$ at N points $x_i^* \in S, i=1..N$

In this case,

$$\psi_i(x) = \delta(x - x_i).$$

Matrix Equations: The Galerkin approximation yields N equations to be solved for $\{B_j\}_j$

$$\begin{aligned} 0 &= \int_S R(x')\,\phi_i(x')\,dA_{x'}, \qquad i = 1..N \\ &= \int_S \left(\sum_{k=1}^N \phi_k(x') \left\{ B_k - E_k - \frac{1}{\pi}\rho_k \sum_{j=1}^N B_j \int_S \phi_j(x)\, V G\, dA_x \right\} \right) \phi_i(x')\, dA_{x'} \\ &= \sum_{k=1}^N \left[(B_k - E_k) \int_S \phi_k(x')\,\phi_i(x')\,dA_{x'} - \frac{1}{\pi}\rho_k \sum_{j=1}^N B_j \int_S dA_{x'}\, \phi_k(x')\,\phi_i(x') \int_S \phi_j(x)\, VG\, dA_x \right] \end{aligned}$$

Constant elements: If we assume constant elements, $\phi_i(x) = \begin{cases} 1, & x \in S_i = \text{triangle "i"} \\ 0, & \text{otherwise} \end{cases}$, we can simplify by using the identities:

$$\begin{cases} \int_S \phi_k\,\phi_i\,dA = \delta_{ik} \cdot \int_{S_i} dA = A_i\,\delta_{ik}. \\ \int_S \phi_i\, g\, dA = \int_{S_i} g\, dA. \quad \text{for an arbitrary } g \text{ function.} \end{cases}$$

$$\Rightarrow 0 = \sum_{k=1}^N \left[(B_k - E_k)\, A_i\, \delta_{ik} - \frac{1}{\pi}\rho_k \sum_{j=1}^N B_j\, \delta_{ik} \left(\int_{S_i} dA_{x'} \int_{S_j} dA_x\, VG \right) \right]$$

Which simplifies to the usual matrix radiosity equations:

$$B_i = E_i + \rho_i \sum_{j=1}^N F_{ij}\, B_j \qquad \text{where } F_{ij} = \frac{1}{A_i} \int_{S_i} \int_{S_j} G(x,x')\, V(x,x')\, dA_x\, dA_{x'}$$