Grothendieck's Inequality:

Given numbers $a_{ij} \in \mathbb{R}$ (both positive or negative) want to figure out how.

$$\text{OPT} := \max_{\substack{x_i \in \{\pm 1\} \\ y_j \in \{\pm 1\}}} \sum a_{ij} x_i y_j$$

relates to

$$\text{SDP} := \max_{\substack{\|u_i\|_2 = 1 \\ \|v_j\|_1 = 1}} \sum a_{ij} \langle u_i, v_j \rangle \quad \longleftarrow \text{the SDP relaxation.}$$

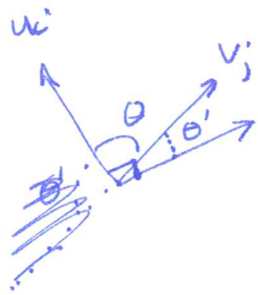
. **Fact**: SDP $\geq$ OPT (it is a relaxation)

Thm: OPT $\geq 0.56 \cdot$ SDP [Krivine, can do better...]

Pf: Suppose we try the simplest idea. These vectors u_i, v_j are on unit ball. So take the random hyperplane, and set $\hat{x}_i = 1$ if $\langle \bar{g} \cdot \bar{u}_i \rangle \geq 0$
$= -1$ if $\langle \bar{g} \cdot \bar{u}_i \rangle < 0$ etc

$$\Rightarrow \hat{x}_i = \text{sign}\langle \bar{g} \cdot \bar{u}_i \rangle, \quad \hat{y}_j = \text{sign}\langle \bar{g} \cdot \bar{v}_j \rangle.$$

then
$$E[\hat{x}_i \hat{y}_j] = -1 \cdot \Pr(ij \text{ separated}) + 1 \cdot \Pr(ij \text{ not separated})$$
$$= -1 \cdot \theta/\pi + 1 \cdot (1 - \theta/\pi) = 1 - \frac{2\theta}{\pi} = \frac{2}{\pi}(\pi/2 - \theta)$$
$$= 2/\pi \cdot \theta'$$



$$\text{SDP}_{ij} = \langle u_i, v_j \rangle = \cos\theta = \sin\theta'$$

$$\cancel{\Rightarrow E[\text{Alg}_{GW}] = E\left[\sum a_{ij} \hat{x}_i \hat{y}_j\right] = \text{SDP} \sum}$$

Here's the problem: for Max Cut we were happy to say that $E[\text{Alg}]_{ij}$ was bigger than SDP_{ij}. But now we have weights that are $\boxed{\text{negative}}$, so cannot do this!

Krivine's Trick:

Lemma: Given $\overbrace{u_1 \dots u_n, v_1 \dots v_m}^{\text{unit}}$, can get $\tilde{u}_i, \tilde{v}_j$ such that

unit vectors

$$\frac{\pi}{2} E\Big[\underbrace{\mathrm{sign}(\langle \tilde{u}_i, g\rangle)}_{\hat{x}_i} \cdot \underbrace{\mathrm{sign}(\langle \tilde{v}_j, g\rangle)}_{\hat{y}_j}\Big] = c \cdot \langle u_i, v_j \rangle$$

equality!!

N.b. $LHS = \frac{\pi}{2} \cdot Alg_{ij} \Rightarrow E[Alg] = E\Big[\sum_{ij}' a_{ij} \hat{x}_i \hat{y}_j\Big] = \sum_{ij}' a_{ij} \cdot \frac{c}{(\pi/2)} \langle u_i, v_j \rangle = \frac{c}{(\pi/2)} \cdot SDP.$

Also: LHS, by calculation above $= \sin^{-1}(\langle \tilde{u}_i, \tilde{v}_j \rangle)$.

So we want to prove that ~~sin~~ $\langle \tilde{u}_i, \tilde{v}_j \rangle = \sin(c \langle u_i, v_j \rangle)$.

existence

Pf: $\sin(c \cdot \langle u_i, v_j \rangle) = \sum_{k=0}^{\infty}{}' (-1)^k \cdot \frac{c^{2k+1}}{(2k+1)!} (u_i \cdot v_j)^{2k+1}$

$$= \sum_{k=0}^{\infty}{}' (-1)^k \cdot \frac{c^{2k+1}}{(2k+1)!} \left(u_i^{\otimes 2k+1} \cdot v_j^{\otimes 2k+1} \right)$$

$(u \cdot v)^2 = u^{\otimes 2} \cdot v^{\otimes 2}$ etc.

So define $\tilde{u}_i$ to be a vector whose k^{th} set of coordinates are $(-1)^k \cdot \sqrt{\frac{c^{2k+1}}{(2k+1)!}} \cdot u_i^{\otimes 2k+1}$

$\tilde{v}_j$ has $\sqrt{\frac{c^{2k+1}}{(2k+1)!}} v_j^{\otimes 2k+1}$

then by def^n $\langle \tilde{u}_i, \tilde{v}_j \rangle = \sin(c \langle u_i, v_j \rangle)$.

What are lengths of $\tilde{u}_i$? $\|\tilde{u}_i\|^2 = \sum_{k=0}^{\infty}{}' \frac{c^{2k+1}}{(2k+1)!} \|u_i\|^{(2k+1)2} = \sinh(c\|u_i\|^2) = \sinh(c)$

So choose $c = \sinh^{-1}(1) = \ln(1+\sqrt{2})$
$= 1.$

similarly $\|\tilde{v}_j\|^2 = \sinh(c) = 1.$

Finally: this was an infinite construction. But this shows $\exists$ vectors that achieve this.

How do you find these vectors? write an SDP for vectors $\tilde{u}_i, \tilde{v}_j$ s.t.

$$\langle \tilde{u}_i \cdot \tilde{v}_j \rangle = \sin(c.\langle u_i, v_j \rangle) \quad \forall i,j$$

inputs!

this SDP gives the solution you want.

<u>Further work</u>: this gave the Grothendieck constant to be $\simeq 0.56$. ← Krivine's bound.

[Alon Makarychev^2 Naor]. Given a general graph, the gap between OPT and SDP

$$OPT = \max_{x_i \in \{\pm 1\}} \sum_{ij \in E}' a_{ij} \langle x_i, x_j \rangle \quad \text{vs} \quad SDP = \max_{\|v_i\|=1} \sum_{ij \in E}' a_{ij} \langle v_i \cdot v_j \rangle$$

same set of vars

is at most $O(\log \vartheta(\bar{G}))$ ← $K(G)$

the Lovasz theta fn on the complement of G.

For bipartite graphs (which are perfect)

$\vartheta(\bar{G}) = \chi(G)$, get constant factor ← we gave explicit bound.

$\leq$ is true in general

(*) They also show that the Krivine bound is <u>not</u> tight, can beat it by a little bit.

<u>Open Question</u>: What is true bound for K_G? Current bounds:-

$$\Omega(\log \omega(G)) \leq K_G \leq O(\log \vartheta(\bar{G}))$$