Grovlen--dicker's inequality:

Given number $a_i \in \mathbb{R}$ (both positive or negative) want to figure out how.

$$\text{OPT} := \max_{x_i \in \pm 1} \sum a_i x_i y_i$$

subject to

$$\text{SDP} := \max \sum a_i \langle u_i, v_j \rangle$$

$\|v_j\|_1 = 1$

Fact: $\text{SDP} \geq \text{OPT}$ (it's a relaxation)

Thm: $\text{OPT} \geq 0.56 \cdot \text{SDP}$

[Krivine, can do better...]

Proof: Suppose we try the simplest idea. These vectors $u_i, v_j$ are on unit ball.

Do take the random hyperplane, and set $\hat{x}_i = 1$ if $\langle g, u_i \rangle \geq 0$

$\hat{x}_i = -1$ if $\langle g, u_i \rangle < 0$

$$\Rightarrow \hat{x}_i = \text{sign} \langle g, u_i \rangle, \quad \hat{y}_j = \text{sign} \langle g, v_j \rangle.$$  

Then $\mathbb{E}[\hat{x}_i \hat{y}_j] = -1 \cdot \Pr(\text{i,j are separated}) + 1 \cdot \Pr(\text{i,j are not separated})$

$$\mathbb{E} = -1 \cdot \frac{\theta}{\pi} + 1 \cdot (1 - \frac{\theta}{\pi}) = 1 - \frac{2 \theta}{\pi} = \frac{2}{\pi} (\frac{\pi}{2} - \theta) = \frac{2 \theta}{\pi}$$

$$\Rightarrow \mathbb{E} = \mathbb{E} = \frac{2 \theta}{\pi}$$

SDP$_{ij} = \langle u_i, v_j \rangle = \cos \theta = \sin \theta'$

Here's the problem: for Max Cut we were happy to say that $\mathbb{E}[A_{ij}]$ was bigger than SDP$_{ij}$. But now we have weights that are negative, so cannot do this!
Krivine's Trick:

Lemma: Given $u_1, \ldots, u_n, v_1, \ldots, v_m$, can we find $u_i, v_j$ such that

$$\frac{1}{2} \text{E} \left[ \sin(\langle u_i, v_j \rangle) \right] \cdot \text{sign}(\langle v_j, u_i \rangle) = c \cdot \langle u_i, v_j \rangle$$

equality!!

Note: LHS = $\frac{1}{2} \cdot \text{Alg}_{i,j} = \text{E} \left[ \text{E} \left[ \sum_{i,j} a_{ij} \langle x_i, y_j \rangle \right] \right] = \sum_{i,j} a_{ij} \cdot \frac{c}{m} \cdot \langle u_i, v_j \rangle = \left( \frac{1}{2} \right) \cdot \text{SDP}.$

Also: LHS, by calculation above = $\sin^{-1}(\langle u_i, v_j \rangle)$.

So we want to prove that $\langle u_i, v_j \rangle = \sin(c \langle u_i, v_j \rangle)$.

Pf: $\sin(c \langle u_i, v_j \rangle) = \sum_{k=0}^{\infty} \frac{(c \langle u_i, v_j \rangle)^{2k+1}}{(2k+1)!} \cdot \langle u_i, v_j \rangle^{2k+1}$

$= \sum_{k=0}^{\infty} (-1)^k \cdot \frac{c^{2k+1}}{(2k+1)!} \cdot \langle u_i, v_j \rangle^{2k+1}$

So define $u_i = \langle u_i \rangle$ to be a vector whose $k$th set of coordinates one $(-1)^k \cdot \sqrt{\frac{c^{2k+1}}{(2k+1)!}} \cdot \langle u_i \rangle$.

$v_j$ has $\sqrt{\frac{c^{2k+1}}{(2k+1)!}} \cdot \langle v_j \rangle$.

Thus by defn $\langle u_i, v_j \rangle = \sin(c \langle u_i, v_j \rangle)$.

What are the $u_i$? $\| u_i \|^2 = \sum_{k=0}^{\infty} \frac{c^{2k+1}}{(2k+1)!} \cdot \| u_i \|^2 = \sinh(c \| u_i \|^2)$

$= \sinh(c)$

So choose $c = \sinh(\theta) = -\ln(1 + \theta^2)$

Similarly $\| v_j \|^2 = \sinh(c) = 1.$

Finally: this was an infinite construction. But this gives 3 vectors that achieve this.
How do you find these vectors? Write an SDP for vectors \( \vec{u}_i, \vec{v}_j \) s.t. 

\[
\langle \vec{u}_i, \vec{v}_j \rangle = \alpha \sin^2 \left( \cos \langle \vec{u}_i, \vec{v}_j \rangle \right) + \gamma
\]

\[\text{[inputs]}\]

This SDP gives the solution you want.

Further work: This gave the Frobenius constant to be \( \alpha = 0.56 \). \(-\) Krivine's bound.

\[\text{[Alon Makarychev, Naor]: Given an even graph, the gap between \( \text{OPT} \) and SDP}
\]

\[
\text{OPT} \quad \quad \sum_{i,j \in E} a_{ij} \langle x_i, x_j \rangle \quad \quad \text{s.t. } \quad \text{SDP} = \max_{\|x\|=1} \sum_{i,j \in E} a_{ij} \langle x_i, x_j \rangle
\]

\[\text{[same set quas]}
\]

is at most \( O(\log \chi(G)) \) \(-\) K(G)

\[\text{[the largest theta-fn on the complement of G].}
\]

For bipartite graphs (which are perfect)

\[
\chi(G) = \chi(G), \quad \text{gave constant factor} \quad \text{[we gave explicit bound].}
\]

\[\text{[there is no invariance.}
\]

(*) They also show that the Krivine bound is not tight, can beat it by a little bit.

Open Question: What is true bound for \( K_\alpha \)? Current bounds:

\[
\Omega(\log \omega(G)) \leq K_\alpha \leq O\left( \log \chi(G) \right)
\]