Max Flow: using MW. (unit capacity graphs).

we want to solve the problem
$$\begin{cases} \sum_P f_P = \cancel{OPT}\ F \leftarrow \text{our guess for the max flow.} \\ \sum'_{P \ni e} f_P \le 1 = u_e \quad \forall \text{edges } e. \\ f_P \ge 0. \end{cases}$$

Given weights $\{p_e\} \in \Delta_m$ we get the problem: find a solution to the problem
$$\begin{cases} \sum' f_P = \cancel{OPT}\ F \\ \sum' p_e \sum'_{P \ni e} f_P \le \sum p_e = 1. \iff \sum'_P f_P \cdot p(P) \le 1 \\ f_P \ge 0. \end{cases}$$

This is an "unconstrained" problem essentially, we should find the shortest path P^* under the edge lengths p_e, and set $f_{P^*} = \cancel{OPT}\ F$. (if $\nexists$ no solution to this problem then $\nexists$ no sol^n to original problem as well.) // this means, length of s.p $\le \frac{1}{OPT}$.

Now once we find this path, we return this sol^n $f_P \longrightarrow$ transformed as an assignment of values $\{f_e\}_{e \in E}$. Note that $f_e \le \cancel{OPT}\ F$ $\forall$ edges. Now we know that

Use one-sided guarantee $\rightarrow$
$$\frac{1}{T}\sum_t \langle f^t, p^t \rangle \ge \frac{1}{T}\sum'_t \langle f^t, \bar{e}_{\text{edge } e_i} \rangle \times(1-\epsilon) - \epsilon \quad \text{for } T \ge \frac{\Theta(\rho \ln m)}{\epsilon^2}$$

If the flow $\hat{f} = \frac{1}{T}\sum f^t$, then RHS $= \max_{e \in E} \langle \hat{f}, e_e \rangle \times(1-\epsilon) - \epsilon$

And $\frac{\sum_t \langle f^t, p^t \rangle}{T} \le 1$. $\Rightarrow$ the edge congestion $\le \frac{(1+\epsilon)}{(1-\epsilon)} \le (1+\epsilon)^2$

Note: Since the ~~losses~~/gains are only positive, we can reduce the dependence on the oracle width (ρ) to ρ instead of ρ^2. [use AHK analysis of MW]

But the runtime now is: $T \times$ shortest path

$$\Rightarrow O\left(\frac{\cancel{OPT}\ F \log m}{\epsilon^2} \times m + n\log n\right) \quad \text{atmost } O(m^2). \quad \text{or } \tilde{O}(m\,\cancel{OPT}\ F)$$

Like FF: Not very impressive! :(But the idea has potential.

②

If only we could find a solution of low width. (ρ). We know $\exists$ an ideal solution (namely the flow itself) that has width $\rho = \max_{e \in E} f_e = 1$.

But ~~that~~ is what we seek!

Here's a way to get width $O(\sqrt{m})$ and still find the flow in time $\tilde{O}(m)$.

Electrical flows!

Suppose we set the voltage (potential) of source s at some value V_s and sink t at $V_t = 0$. then you get electrical current through the network that minimizes the total energy dissipated, which is $\sum f_e^2 . R_e$, where R_e is resistance of the edge. Recall also that $\Delta\phi$ across the edge $= f_e . R_e$.

$$\text{Energy} = \sum_e f_e^2 R_e = \sum_{u \sim v} \frac{(\phi_u - \phi_v)^2}{R_e}.$$

How do we find this flow :-

L = Laplacian of the graph = Diag(degrees) − A. (unweighted).

Note: Laplacian of an edge $(u,v) = \begin{array}{c|cc} & u & v \\ \hline u & 1 & -1 \\ v & -1 & 1 \end{array}$ or with resistance R_e $\left(\quad\right) \times \frac{1}{R_e}$.

$$\text{Laplacian of graph} = \sum_{\text{edges}} \text{Laplacian of the edge} \cdot \left(\frac{1}{R_e}\right).$$

$$\text{So } Lx = \sum_{u \sim v} \frac{L_{uv} . x}{R_e} \cancel{\sum_{u\sim v} \frac{(x_u - x_v)^2}{R_e}} = \left(\sum_{v \sim u} \frac{(x_u - x_v)}{R_e}\right)_{u \in U}$$

$(Lx)_u$: this should be total current being pushed into u.

So if we set $(Lx)_s = OPT = -(Lx)_t$ And solve for x, we get the potentials.
$(Lx)_u = 0 \ \forall u \notin \{s, t\}$

And hence the f_e s.

Good: the best part is: we can solve $Lx=b$ for Laplacian matrices in time $O(m\,\mathrm{poly}\lg n)$. [Spielman Teng, Miller Koutis Peng ....] In fact in time $O(m\sqrt{\lg n})$ now. [Check: is linear doable already?].

Why is this Electrical flow better than just the shortest path flow?

Intuitively, does not overload any edge, since the energy dissipated is f_e^2.

So will try to spread it out a little more

Formally: Suppose we set resistances $R_e = \cancel{\tfrac{1}{\epsilon}}\, p_e + \frac{\epsilon}{m}$ then get electrical flow f_e

with
- $\sum_i p_e f_e \le (1+\epsilon)$ ← instead of 1, but no real big deal.
- $\max_e f_e \le \tilde{O}(\sqrt{m})$. ← width!

Pf: Since $\exists$ a feasible flow f^* with $f^*(e) \le 1 \ \forall e$, it has energy at most $\text{Energy}(f^*) \le \sum_i R_e \cdot (f_e^*)^2 \le \cancel{(1+\epsilon)} \sum_i (p_e + \epsilon/m)$

$\le 1+\epsilon$.

$$\Rightarrow \left(\sum p_e f_e\right)^2 \le \left(\sum_i p_e f_e^2\right)\left(\sum_i p_e\right) \le (1+\epsilon).$$

$$\Rightarrow \sum p_e f_e \le \sqrt{(1+\epsilon)} \ \cancel{= 1 + O(\epsilon)} \le (1+\epsilon) \ \checkmark$$

also $\forall e$ $\quad \frac{f_e^2 \cdot \epsilon}{m}$ ← at least this much energy burn on edge e $\quad \le (1+\epsilon)$ total energy burn on min energy flow

$$\Rightarrow f_e \le \sqrt{\frac{m(1+\epsilon)}{\epsilon}} \approx \sqrt{m}.$$

Plugging this in, get an $\tilde{O}(m^{3/2})$ runtime algorithm.

Even for sparse graphs, ~~not~~ as good as Even-Tarjan but no better.

How can we do better?

[Christiano Kelner Madry Spielman Teng 2011].

One Sided Guarantee:

[Arora Hazan Kale Thm 2.1]

Thm 1: $\forall \epsilon \le \frac{1}{2}$, The MW($\epsilon$) algorithm achieves the following guarantee:-
for any sequence of loss vectors $\ell^t \in [-1,1]^N$, $\forall T$, $\forall i$,

$$\sum_t \langle \ell^t, p^t \rangle \le \sum_t \langle \ell^t, e_i \rangle + \epsilon \sum_t |\ell_i^t| + \frac{\ln N}{\epsilon}.$$

Now translating to gains, and scaling by ρ, (i.e. setting $\ell^t = -\frac{g^t}{\rho}$) we get

Thm 2: $\forall \epsilon \le 1/2$, the MW(ϵ) algorithm achieves, for all sequence of gain vectors $g^t \in [-\rho, \rho]^N$, $\forall T$, $\forall i$

$$\sum_t \langle g^t, p^t \rangle \ge \sum_t g_i^t - \epsilon \sum_t |g_i^t| - \rho \frac{\ln N}{\epsilon} \qquad (*)$$

What if the gains are only positive? in $[0, \rho]$ then $(*)$ simplifies to

$$\sum_t \langle g^t, p^t \rangle \ge \left(\sum_t g_i^t\right)(1-\epsilon) - \frac{\rho \ln N}{\epsilon}$$

Now setting T large enough, we get that (i.e. $T \ge \frac{\rho \ln N}{\epsilon^2}$)

$$\text{Average gain} = \frac{1}{T} \sum_t \langle g^t, p^t \rangle$$

$$\ge \frac{(\text{Gain of expert}_i)}{T}(1-\epsilon) - \left(\frac{\rho \ln N}{\epsilon} \cdot \frac{1}{T}\right)$$

$$= (\text{Average gain of expert } i)(1-\epsilon) - \epsilon$$

■