Polyhedron: intersection of some finite # of halfspaces $a_i^T x \geq b_i$ (say)
in $\mathbb{R}^n$

$$K = \{x \in \mathbb{R}^n \mid Ax \geq b\}.$$

Polytope: bounded polyhedron, sits within some ball $B(0,R)$.

LP: $\min\{c^T x \mid Ax \geq b, ~~x \geq 0~~\}$ $= \min\{c^T x \mid x \in K\}$

~~Thm~~ Fact: if K is a polytope then ∃ an optimum achieved at an extreme pt of K.

- What is extreme pt?

  (*) x is extreme if $x = \lambda x_1 + (1-\lambda) x_2 \Rightarrow x_1 = x_2 = x$.
  $x \in K$ $\quad x_1, x_2 \in K$

- Vertex of K: x is a vertex of K if $\exists \tilde{c}$ st $\tilde{c}^T x < \tilde{c}^T y \;\; \forall y \neq x$ in K.

- Basic feasible solution: x^* is a BFS if (a) x^* is feasible $x \in K$
  and (b) $\exists\, n$ Linearly Indep constraints $a_i^T x \geq b_i$ which are tight at x^* $\boxed{a_i^T x = b_i}$

  they form a "basis" and x^* is the unique intersection of those n LI constraints.

~~Thm: Sps K is a polytope. then the optimum point of LP = $\min\{c^T x \mid x \in K\}$ is achieved at~~

Fact: All three definitions are identical. BFS $\Leftrightarrow$ vertex $\Leftrightarrow$ extreme point.
(~~for K being a polytope~~)
~~at least~~.

$\Rightarrow$ optimum achieved at BFS $\Leftrightarrow$ vertex $\Leftrightarrow$ extreme pt.

In general, not: imagine $\min\{x_1 + x_2 ;\; x_1 + x_2 \geq 2\}$.

An integer polytope is one where all vertices are in $\mathbb{Z}^n$.

We're interested in polytopes that capture combinatorial objects, $\subseteq [0,1]^n$

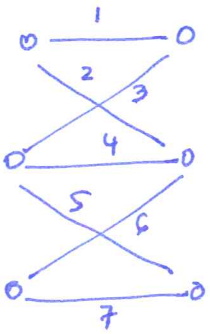
⇒ here, all vertices are in $\{0,1\}^n$.

Fact: two ways to define polytope ← bounded, remember

① $Ax \geq b$.

② if $\{v_1, v_2 \dots v_N\}$ are the vertices then $CH(v_1 \dots v_N)$.

OK: lets look at perfect matchings of a bipartite graph G. write these as vectors in space $\{0,1\}^m$



$\Rightarrow$ (1001001)
(0110001)
(1000110) are all the PMs.

⇒ Perfect matching polytope = CH of these. CH_{PM}

Why? Want max wt PM? max $w^T x$ st $x \in CH_{PM}$.
basic sol^n will find a single matching that maximizes the wt.

⇒ perfect.

But: this is a painful way to specify CH_{PM}. We have to write CH (exponential # of vertices)

Nicer way using $\{Ax \geq b\}$? Yup.

Claim: $K_{PM} := \{ \sum_j x_{ij} = 1 \ \forall i, \ \sum_i x_{ij} = 1 \ \forall j \}$.

then $K_{PM} = CH_{PM}$

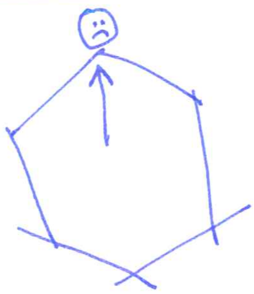
Proof 1 : • $CH_{PM} \subseteq K_{PM}$ since each PM satisfies constraints.

• Every vertex of K_{PM} is in CH_{PM}. Why?

Last time we showed an algorithm that given weights w would find an integer matching that was optimal for the LP $\{\max_{x \in} w^T x \mid K_{PM}\}$. [we showed LP gave primal = dual ⇒ optimal].

⇒ the optimal point ~~of~~ within K_{PM} (for w) $\in CH_{PM}$.
"vertex"

⇒ all vertices of $K_{PM} \in CH_{PM}$ ⇒ $K_{PM} \subseteq CH_{PM}$.



"If K_{PM} had a non integer vertex, we ~~could~~ could give it that weight as objective and algorithm would fail".

Proof 2 : (using extreme points)

~~[illegible]~~ take any extreme pt of K_{PM}.

if cycle can write as convex comb. ⇒ no cycle
⇓
~~leaf~~ ⇒ just edges.

Proof 3 (using BFS).

take any BFS of K_{PM}.

must have ~~[illegible]~~ m tight constraints that are LI.

where are they?

etc. [see previous lecture notes].

Can you use
Same idea to show that

$$K_{Arb} = \left\{ \begin{array}{l} x(\partial^+ v) = 1 \\ x(\partial^+ S) \geq 1 \quad \forall S,\ r \notin S,\ S \neq \emptyset \\ x \geq 0 \end{array} \right\} \Rightarrow CH_{Arborescences}.$$

$$K_{MST} = \left\{ \begin{array}{l} x(\partial S) \geq 1 \quad \forall S \neq \emptyset \\ x\left(\binom{S}{2}\right) \leq |S| - 1 \\ x\left(\binom{n}{2}\right) = n - 1 \\ x \geq 0 \end{array} \right\}.$$

Non Bipartite Matching:

$$K_{PM_{gen}} = \{ x : x(\partial v) = 1,\ x(\partial S) \geq 1 \ \forall S \text{ odd},\ x \geq 0 \}.$$

Claim: $K_{PM_{gen}} = CH(\text{all perfect matchings in } G)$

Let's prove using rank ~~op~~ (lofs) approach. [See previous notes]