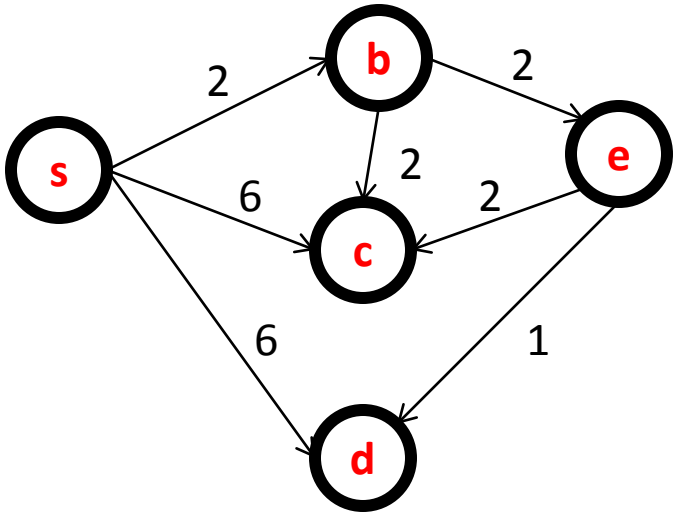
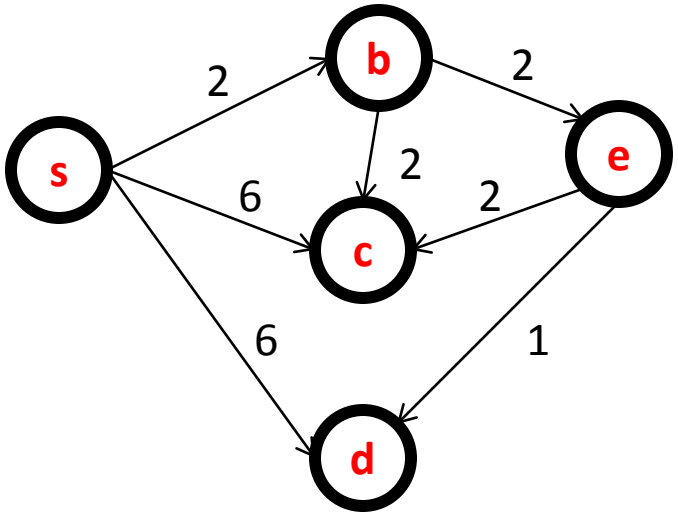
Dijkstra's Algorithm
SSSP, non-neg



Edge weights = w(x,y)

Final distances = d(x,y) = $d_w(x,y)$

Dijkstra's Algorithm
SSSP, non-neg



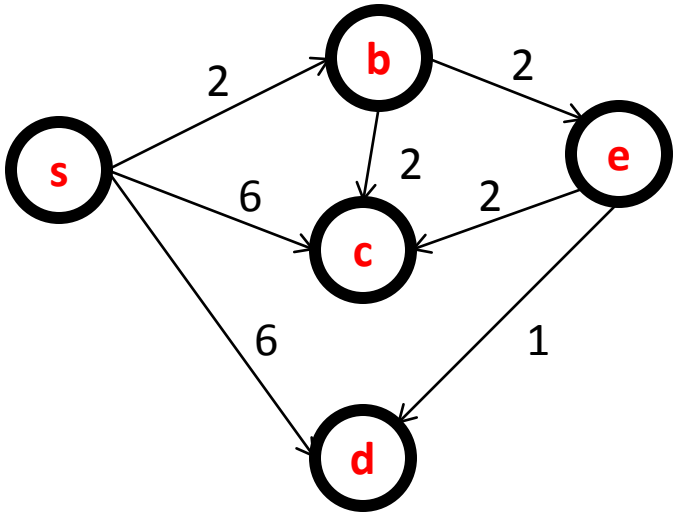
	s	b	c	d	e
L(x)	0	∞	∞	∞	∞

x <- extractmin

// L(x) is the distance of s to x
// mark x as final

"relax" all the edges out of x
L(y) <- min (L(y), L(x) + w(x,y))

Dijkstra's Algorithm
SSSP, non-neg



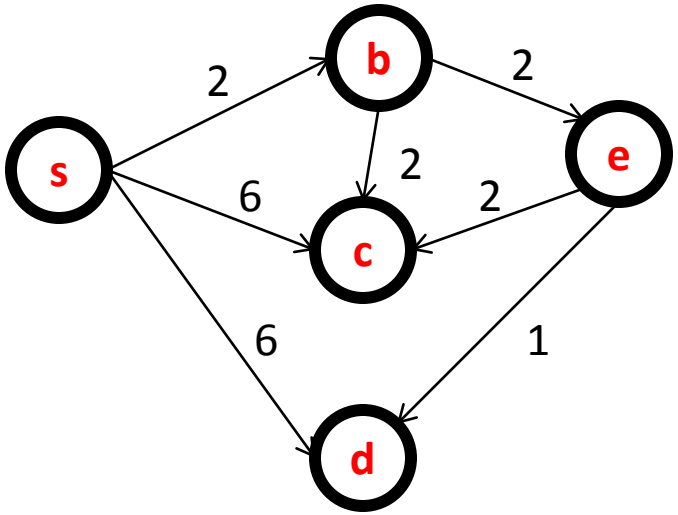
	s	b	c	d	e
L(x)	0	2	6	6	∞

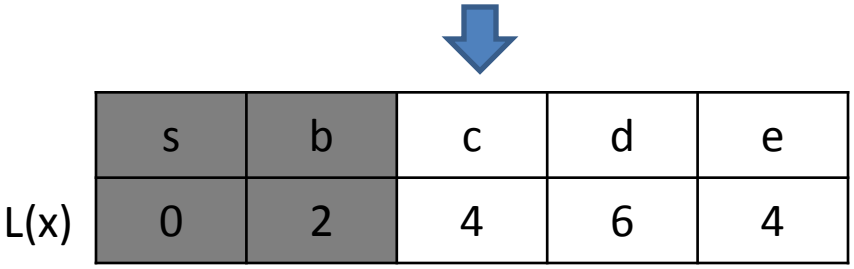
x <- extractmin

// L(x) is the distance of s to x
// mark x as final

"relax" all the edges out of x
L(y) <- min (L(y), L(x) + w(x,y))

Dijkstra's Algorithm
SSSP, non-neg





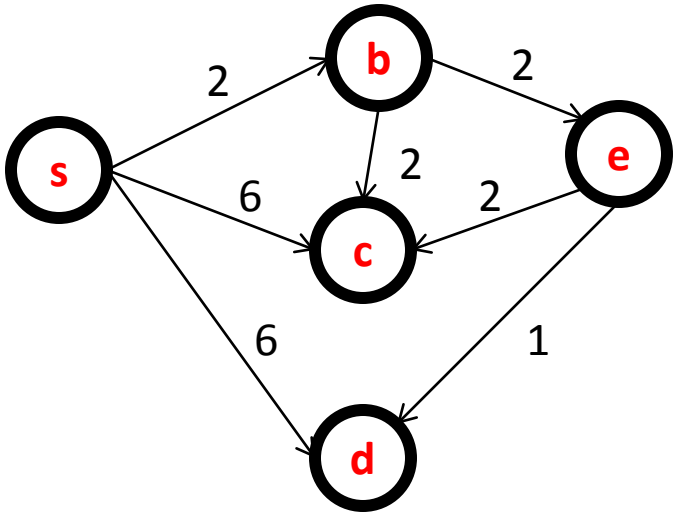
	s	b	c	d	e
L(x)	0	2	4	6	4

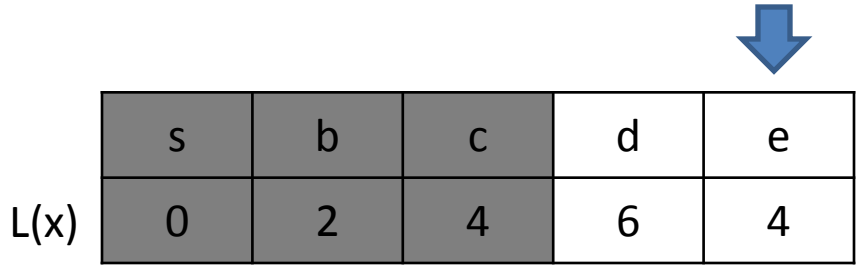
x <- extractmin

// L(x) is the distance of s to x
// mark x as final

"relax" all the edges out of x
L(y) <- min (L(y), L(x) + w(x,y))

Dijkstra's Algorithm
SSSP, non-neg





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L(x)	0	2	4	6	4

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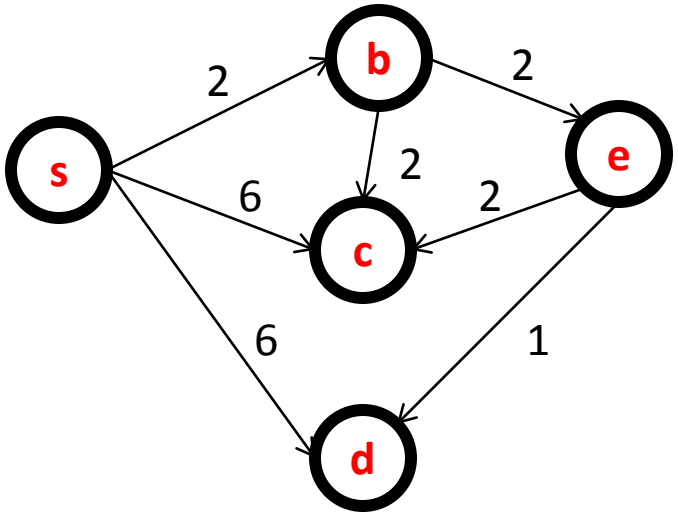
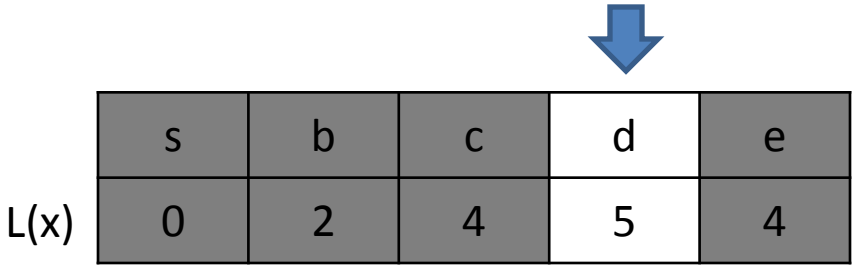
Dijkstra's Algorithm
SSSP, non-neg





	s	b	c	d	e
L(x)	0	2	4	5	4

x <- extractmin

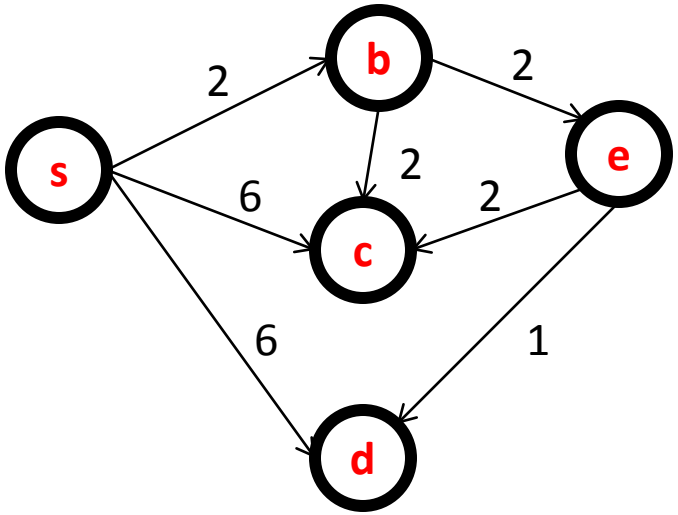
// L(x) is the distance of s to x
// mark x as final

"relax" all the edges out of x
L(y) <- min (L(y), L(x) + w(x,y))

Dijkstra's Algorithm
SSSP, non-neg

m decreasekeys
n extractmins

Fib heap: O(m + n log n)



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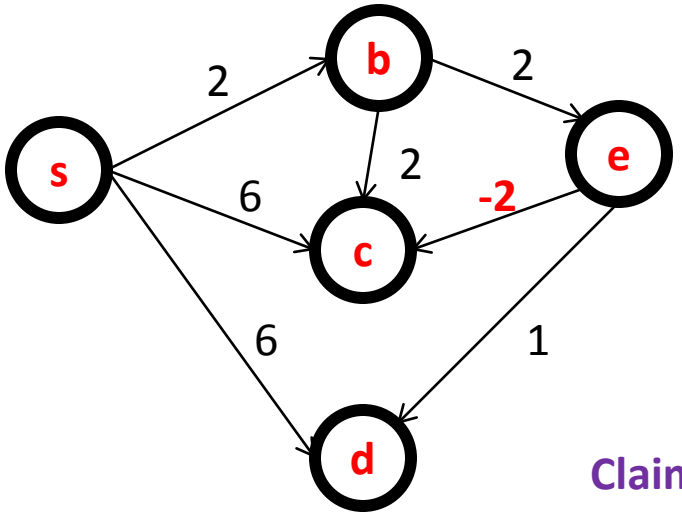
Bellman-Ford-Moore Algorithm
SSSP, neg wts OK

Bellman-Ford-Moore Algorithm
SSSP, neg wts OK

n rounds
m time per round
O(mn) time

	s	b	c	d	e
L(x)	0	∞	∞	∞	∞



// L(x) is an upper bound on d(s,x)

for t = 1 to n-1
  for all vertices x
    // "relax" all the edges out of x
    L(y) <- min (L(y), L(x) + w(x,y))

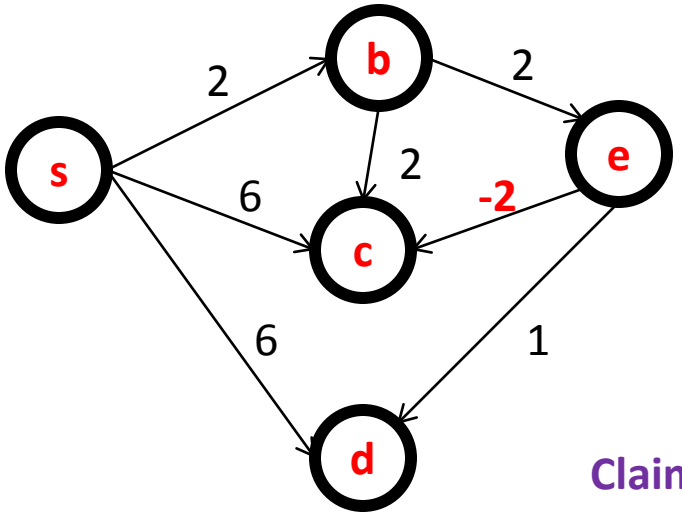
Claim: if graph has non negative cycles, B-F-M is OK.

Proof: induction. At end of round t, L(y) is shortest path using at most t edges.

Bellman-Ford-Moore Algorithm
SSSP, neg wts OK

n rounds
m time per round
O(mn) time

	s	b	c	d	e
L(x)	0	2	**2**	5	4

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```

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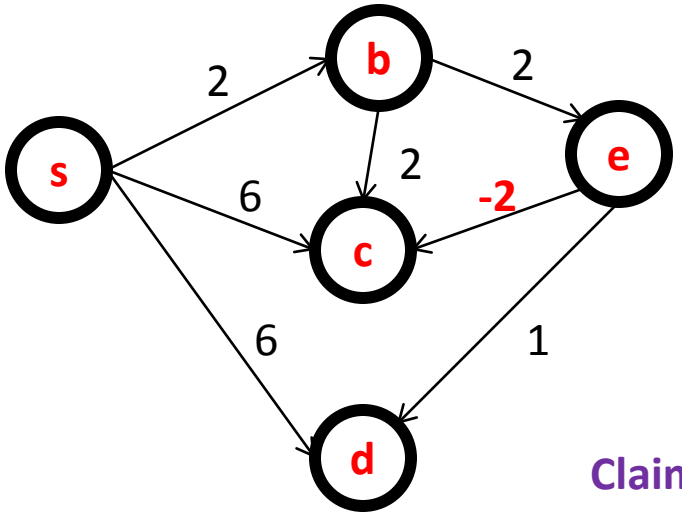



```
// L(x) is an upper bound on w(s,x)

for t = 1 to n-1
    for all vertices x
        // "relax" all the edges out of x
        L(y) <- min ( L(y),  L(x) + w(x,y) )
```

Claim: If at end, some edge xy is "over-tight"
(it has L(y) > L(x) + w(x,y))
then graph has negative cycle.

Lots more work!

E.g., extensions and implementations of (just) Dijkstra's algorithm:

$O(m + n^2)$	Dijkstra'59
$O(m \log n)$	William'64
$O(m + n \log n)$	Fredman and Tarjan'87
$O(m\sqrt{\log n})$	Fredman and Willard'93
$O(m + n\frac{\log n}{\log \log n})$	Fredman and Willard'94
$O(m \log \log n)$	Thorup'96
$O(m + n\sqrt{\log n}^{1+\varepsilon})$	Thorup'96
$O(m + n\sqrt[3]{\log n}^{1+\varepsilon})$	Raman'97
$O(m + n\sqrt[3]{\log n}^{1+\varepsilon})$	Raman'97
$O(m\sqrt{\log \log n})$	Han and Thorup'02
$O(m + n \log \log n)$	Thorup'03

$O(m \log \log C)$	van Emde Boas'77
$O(m + n\sqrt{\log C})$	Ahuja et.al.'90
$O(m + n\sqrt[3]{\log C \log \log C})$	Cherkassky et.al.'97
$O(m + n\sqrt[4]{\log C}^{1+\varepsilon})$	Raman'97
$O(m + n \log \log C)$	Thorup'03

slide from Mikkel Thorup

All Pairs SP

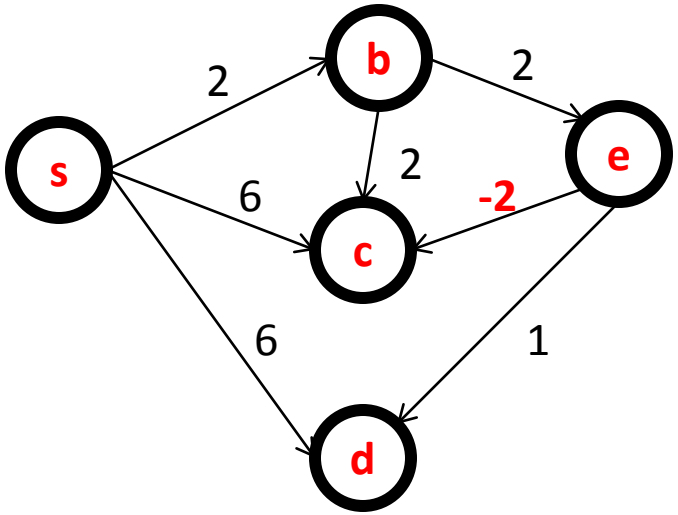
Non-neg weights:

n Dijkstras — $O(mn + n^2 \log n)$

Neg weights:

n B-F-M — $O(mn^2)$

Johnson's Algorithm
APSP, neg wts OK



Find "feasible potentials" ©(x) such that the "reduced weights"

$\hat{w}(x,y) := ©(x) + w(x,y) - ©(y) \geq 0$

Fact: reduced weights $\hat{w}$ non-neg

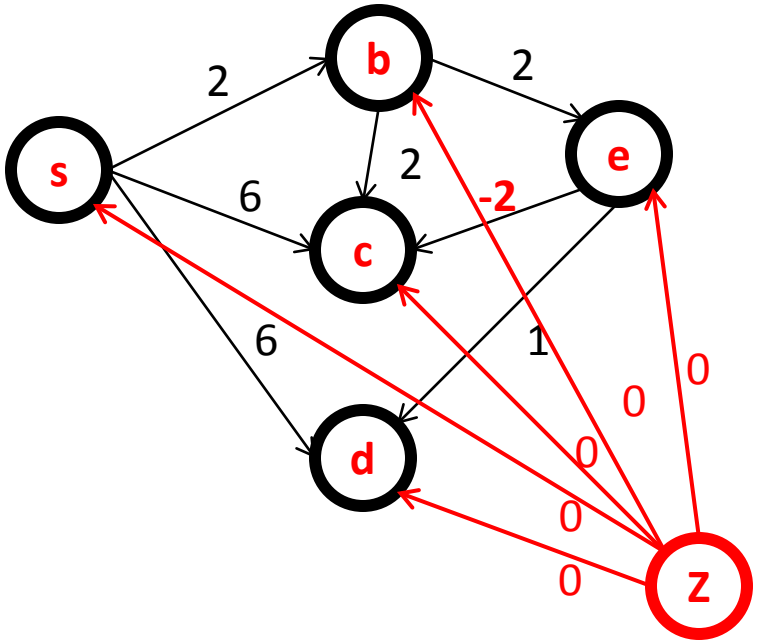
Claim: $d_w(x,y) = d_{\hat{w}}(x,y) + ©(y) - ©(x)$

So shortest paths don't change, though their weights might.

$\Rightarrow$ suffices to find APSP in this non-neg weights graph!

How to find these "feasible potentials"?

Johnson's Algorithm
APSP, neg wts OK



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So shortest paths don't change, though their weights might.

$\Rightarrow$ suffices to find APSP in this non-neg weights graph!

How to find these "feasible potentials"?

Shortest paths from Z.

No negative cycles created, so B-F-M works.

All Pairs SP

Non-neg weights:
n Dijkstras $O(mn + n^2 \log n)$

Neg weights:
n B-F-M $O(mn^2)$

Neg weights:
B-F-M + n Dijkstras $O(mn + n^2 \log n)$

Neg weights:
Floyd-Warshall $O(n^3)$

```
for all vertices z
    for all vertices x,y
        d(x,y) <- min{ d(x,y), d(x,z) + d(z,y)
```

All Pairs SP

Non-neg weights:
n Dijkstras $O(mn + n^2 \log n)$

Neg weights:
n B-F-M $O(mn^2)$

Neg weights:
B-F-M + n Dijkstras $O(mn + n^2 \log n)$

Neg weights:
Floyd-Warshall $O(n^3)$

Neg weights:
Naïve Min-Sum-Product $O(n^3 \log n)$