15-859FF: Coping with Intractability | **CMU, Fall 2019**
Lecture #21: Recent Advances in Hardness Results | November 13, 2018
Lecturer: Venkatesan Guruswami | Scribe: Max Bender

1 Problematic Problems

Around the year 2008, the following four problems had remained unresolved for some time; that is, for each general[1] problem there had been no FPT algorithm established nor any proof of hardness:

Bipartite Clique (BiClique):
Input: A biparitite graph $G = (L, R, E)$ and integer k
Parameter: k
Question: Does G contain a complete bipartite graph $K_{k,k}$ as a subgraph?

GraphMulticut:
Input: A graph $G = (V, E)$ and pairs of vertices $\{\{s_i, t_i\} \mid 1 \le i \le m\}$
Parameter: k
Question: Is there a set $S \subseteq V$ with $|S| = k$ so that in the graph $G[V \setminus S]$ each pair of vertices $\{s_i, t_i\}$ lie in different components?

Directed Feedback Vertex Set (DFVS):
Input: A directed graph $G = (V, E)$
Parameter: k
Question: Is there a set $S \subseteq V$ with $|S| = k$ so that $G[V \setminus S]$ is acyclic?

EvenSet:
Input: A matrix $A \in \mathbb{F}_2^{n \times n}$
Parameter: k
Question: Is there an $x \in \mathbb{F}_2^n$ so that $Ax = 0$ and x has at most k nonzero coordinates?

In fact, even by 2013 only GraphMulticut had been shown to be in FPT (see [MR10] and [BDT10]), while the other three still remained unresolved[2]. Of BiClique in particular, [DF13] wrote: "Almost everyone considers that this problem should obviously be $\mathbf{W}[1]$-hard... It is rather an embarassment to the field that the question remains open after all these years!" Fortunately, there has been significant progress in recent years, with [Lin14] showing that BiClique is in fact $\mathbf{W}[1]$-hard and [BdBE$^+$19] showing EvenSet is $\mathbf{W}[1]$-hard. These notes will be focused on the result that BiClique is $\mathbf{W}[1]$-hard. To prove this result, we will show that Clique $\le_{FPT}$ BiClique, since Clique is already known to be $\mathbf{W}[1]$-hard.

2 The Bad Reduction

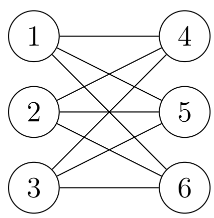
First, let's consider the "intuitive" but unfortunately incorrect reduction. Given a graph $G = (V, E)$ and parameter k as an instance for Clique, we create an instance $G' = (L, R, E')$ for BiClique with the same parameter k. To do so, we will create a copy of V denoted $V' = \{v' \mid v \in V\}$ and set

[1] advancements had been made under various assumptions

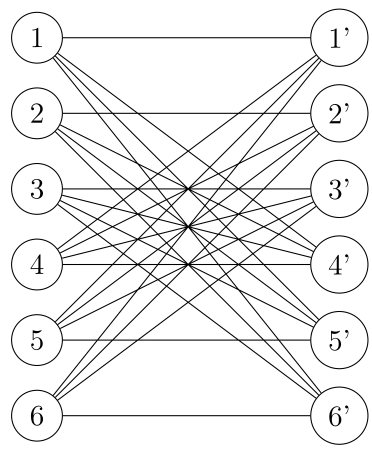
[2] DFVS was shown to be in FPT in [CLL$^+$08], however it was still unknown whether a kernelization linear in k could be found

$L = V$ and $R = V'$. To create the edges, we make an edge between any vertex in V and its copy, as well as maintaining all edges we had previously but only going between the two vertex sets L and R to maintain bipartiteness. That is, we define $E' = \{\{v, v'\} \mid v \in V\} \cup \{\{u, v'\} \mid \{u, v\} \in E\}$.

Now, it would be really convenient if there were a clique of size k in G if and only if $K_{k,k}$ were a subgraph of G', since this would in fact be a polytime reduction and hence would be sufficient to show that CLIQUE $\leq_{FPT}$ BICLIQUE (moreover, that CLIQUE $\leq_p$ BICLIQUE). Unfortunately, this is not the case. While a clique in G certainly induces a biclique in G' of the same size, the reverse does not hold. Consider the case where G is actually $K_{n,n}$: then for $n > 2$, G does not have a clique of size n. However, G' will contain $K_{n,n}$ as a subgraph. Hence this isn't a valid reduction. The case for $G = K_{3,3}$ is drawn below:



G



G'

Note that G has no clique of size 3, while G' has $K_{3,3}$ as a subgraph (consider $\{1, 2, 3, 4', 5', 6'\}$).

3 A better idea

Our first idea for a reduction may have failed, but still we must persevere! The idea for this reduction will be to use a 'gadget', or what we will refer to as The Magic Graph.

3.1 The Magic Graph

What we want is a bipartite graph that somehow makes sure that every large subset of the left vertices can't be involved in a bipartite clique of the same size, while every small subset of vertices on the left side induces a bipartite clique of the same size. Suppose we had a graph $T = (L = [n], R = [n], E)$ satisfying the following two properties:

T1) For all $S \subseteq L$ with $|S| = k + 1$, we have that $|C_N(S)| \leq \ell$

T2) For all $S \subseteq L$ with $|S| = k$, we have that $|C_N(S)| \geq h$

where $C_N(S) = \{v \in R \mid \text{there exists a } u \in S \text{ so that } \{u, v\} \in E\}$ and $\ell < h$.

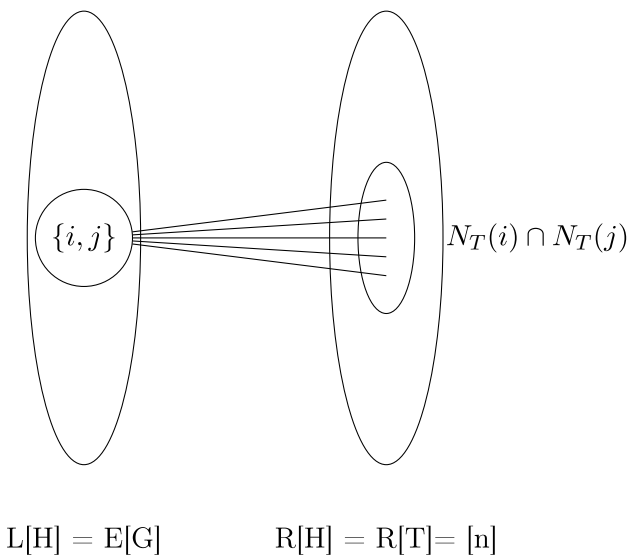
First, consider the following problem:

NONEXACTBICLIQUE *Input*: A bipartite graph $G = (L, R, E)$ and integers $s \leq \ell < h$ *Output*: Accept if $K_{s,h}$ is a subgraph of G, and reject if $K_{s,\ell+1}$ is not a subgraph of G.

Note that this problem can be reduced to determining whether $K_{h,h}$ is a subgraph of G by adding $h-\ell-1$ vertices to R and making them adjacent to all vertices of L, and adding $h-s$ vertices to the left and making them all adjacent to vertices in the right. Hence if $K_{s,\ell}$ were the largest biclique with s vertices from the left, then $K_{s+(h-s),\ell+(h-\ell-1)} = K_{h,h-1}$ would be the largest in the new graph, while a biclique of size $K_{s,\ell+1}$ would induce a biclique of size $K_{h,h}$. Hence it would be sufficient to reduce CLIQUE to this problem for any suitable choices of s, ℓ, and h.

3.2 Reduction via Magic Graph

In order to reduce CLIQUE to NONEXACTBICLIQUE, we will use any graph $T = (L[T] = [n], R[T] = [n], E[T])$ satisfying the above properties T1 and T2. That is, given $G = (V = [n], E[G])$ and parameter k, we create the graph $H = (L[H], R[H], E[H])$ so that $L[H] = E[G]$, and $R[H] = R[T]$. To create $E[H]$, we will use the following rule: we create an edge between $\{i,j\} \in E[G] = L[H]$ and $w \in R[H]$ if $\{i,w\}, \{j,w\} \in E[T]$. In another words, letting $N_G(u) = \{v \mid \{u,v\} \in E[G]\}$ (ie, $N_G(u)$ is the neighborhood of u in G), we set $N_H(\{i,j\}) = N_T(i) \cap N_T(j)$.



Claim: G has a k-Clique if and only if H has a subgraph $K_{s,h}$ where $s = \binom{k}{2}$.

Proof. If G has a k-Clique, then let $X \subseteq V = [n]$ be the k vertices and let S be the edges incident to X, ie $S = \{\{u,v\} \mid u \in X\}$. Since X is a k-Clique, we have $|S| = \binom{k}{2} = s$. Since $|X| = k$, by T2 we have that $|C_N(X)| \geq h$ in T, so

$$\cap_{e=\{i,j\}\in S} N_G(e) = \cap_{\{i,j\}\in S} N_T(i) \cap N_T(j) = \cap_{x\in X} N_T(i) = C_N(X).$$

Thus taking S and all their common neighbors in H forms a bipartite clique of size at least $K_{s,h}$.

If G has no k-Clique, then let $S \subseteq E[G]$ with $|S| \geq s = \binom{k}{2}$ with $X \subseteq V$ being defined as the set of vertices incident to S in G. Since G has no k-Clique, it must be that $|X| \geq k+1$. Hence by T1 we know that $|C_N(X)| \leq \ell$. Since

$$\cap_{e=\{i,j\}\in S} N_G(e) = \cap_{\{i,j\}\in S} N_T(i) \cap N_T(j) = \cap_{x\in X} N_T(i) = C_N(X),$$

this tells us that any set S of size at least s must have no more than ℓ common neighbors. Since $\ell < h$, this gives us that there is no $K_{s,h}$ in H. $\square$

This immediately tells us that if we can construct such a graph T for choices of h and ℓ as functions of k, we can create the reduction.

3.3 The Good News and the Bad News

The bad news is that we do not know how to construct such a magic graph T, or even show its existence via probabilistic methods.

However, the good news is that one can relax property (T2) to (T2') in a manner so that the reduction still works, and one can construct a magic graph meeting this relaxed property (T2') (along with (T1) which we will keep as is). We won't get into the specifics of (T2'), but we will just mention that it can be constructed for any given k, n with $\ell = (k+6)!$ and $h = n^{\Theta(\frac{1}{k})}$.

The construction builds on the bipartite graph T with left and right hand sides $L[T] = R[T] = \mathcal{F}_q$, where $\mathcal{F}_q$ is the finite field with q elements. It creates edges between $x \in L[T]$ and $y \in R[T]$ if $(x+y)^{\frac{q-1}{d}} = 1$ for an appropriate choice of d. By doing so, you get the property (T1), see [BGK+96]. The construction is modified in [Lin14] to also get (T2').

However, we can give a simpler reduction using randomization! These details will be covered more precisely in the next lecture, but as a warmup consider our two properties T1 and T2 in the setting of random graphs. Generally speaking, by adding edges independently with some constant probability, it will be relatively easy to guarantee that general sets of vertices can't have too many common neighbors. Formally, consider a random bipartite graph $T = ([n], [n], E)$ where each edge $\{i, j\}$ exists with probability p. We will show that (T1) holds with high probability, i.e., with high probability, there is no $K_{k+1,\ell+1}$ subgraph of T. To do this, we will show that the expected number of such subgraphs will be much less than 1. Note that the number of such subgraphs is just the number of ways of choosing $k+1$ vertices from the left and $\ell+1$ of the right, against the probability that every edge is present between the two sets chosen; that is,

$$\begin{aligned} \mathbf{E}[\# \text{ of } K_{k+1,\ell+1} \text{ in } T] &= \binom{n}{k+1}\binom{n}{\ell+1} p^{(k+1)(\ell+1)} \\ &\leq n^{k+\ell+2} - \theta(k+1)(\ell+1) \leq \frac{1}{n} \end{aligned}$$

when $p = n^{-\theta}$ and $\theta = \frac{k+\ell+3}{(k+1)(\ell+1)}$.

For (T2), we note that we only care about it holding (with high probability) when the left hand side is the specific k-clique in the Clique instance. We do not need it to hold for every set of k vertices on the left. We will show this happens w.h.p. by computing the expected number of $K_{k,h}$ with a fixed left set of k vertices, and then using a second moment computation to argue this number is positive with high probability. The details will be presented in the next lecture.

References

[BdBE+19] Arnab Bhattacharyya, ÃĽdouard Bonnet, LÃąszlÃş Egri, Suprovat Ghoshal, Karthik C. S., Bingkai Lin, Pasin Manurangsi, and DÃąniel Marx. Parameterized intractability of even set and shortest vector problem, 2019. 1

[BDT10] Nicolas Bousquet, Jean Daligault, and StÃľphan ThomassÃľ. Multicut is fpt, 2010. 1

[BGK+96] László Babai, Anna Gál, János Kollár, Lajos Rónyai, Tibor Szabó, and Avi Wigderson. Extremal bipartite graphs and superpolynomial lower bounds for monotone span programs. In *STOC*, 1996. 3.3

[CLL+08] Jianer Chen, Yang Liu, Songjian Lu, Barry O'Sullivan, and Igor Razgon. A fixed-parameter algorithm for the directed feedback vertex set problem. *J. ACM*, 55, 10 2008. 2

[DF13] Rodney G. Downey and Michael R. Fellows. *Fundamentals of Parameterized Complexity*. Springer Publishing Company, Incorporated, 2013. 1

[Lin14] Bingkai Lin. The parameterized complexity of k-biclique, 2014. 1, 3.3

[MR10] DÃąniel Marx and Igor Razgon. Fixed-parameter tractability of multicut parameterized by the size of the cutset, 2010. 1