15-859FF: Coping with Intractability **CMU, Fall 2019**
Lecture #15: New Proof of the Planar Separator Theorem October 10, 2019
Lecturer: Jason Li Scribe: Hoon Oh

1 Planar Separator Theorem

Theorem 15.1 (Planar Separator Theorem [LT79])**.** *For any planar graph $G = (V, E)$ on $n = |V|$ vertices, we can remove $O(\sqrt{n})$ vertices such that every connected component in the remaining graph has at most $\frac{2n}{3}$ vertices.*

In the previous lecture, we saw a combinatorial proof of the Theorem 15.1. In this lecture, we will show an algorithmic proof of the theorem. Throughout the notes, n denotes the number of vertices. Let $F(G)$, $V(G)$, and $E(G)$ be the set of faces, the set of vertices, and the set of edges of G respectively.

Definition 15.2 (Fundamental cycle)**.** Given a spanning tree $T \subseteq G$ of a planar graph, a *fundamental cycle* is an edge $(u, v) \notin T$ together with the path from u to v on the tree T.

We prove the following lemma in Section 3.

Lemma 15.3. *Given a spanning tree $T \subseteq G$ of a triangulated planar graph, there is a fundamental cycle C such that $|\mathsf{In}(C)| \leq \frac{2n}{3}$ and $|\mathsf{Out}(C)| \leq \frac{2n}{3}$.*

Given a planar representation of the graph G, $\mathsf{In}(C)$ is the set of vertices strictly inside the cycle C in G, and $\mathsf{Out}(C)$ is the set of vertices strictly outside of the cycle C in G.

Corollary 15.4. *Any planar graph G has a separator of size at most $2\mathsf{diam}(G) + 1$, where $\mathsf{diam}(G)$ is the diameter of a graph G.*

Proof. We first triangulate the graph G. Then consider a BFS tree T of G. Note the depth of the tree is at most $\mathsf{diam}(G)$. Therefore, any fundamental cycle of T has length at most $2\mathsf{diam}(G) + 1$. Furthermore, the cycle is a separator by Lemma 15.3. □

Therefore, if G has a diameter of $O(\sqrt{n})$ then Lemma 15.3 implies Theorem 15.1. Before proving Lemma 15.3, we will introduce the notion of dual of a planar graph. The dual graph will play a crucial role in our algorithm.

2 Dual of a Planar Graph

Definition 15.5 (Dual of a planar graph)**.** Given a planar graph G, the dual graph G^* has vertices as faces of G and there's an edge between two faces in G^* if they are adjacent in G.

Figure 15.1 shows a planar graph and its dual graph. Black vertices and black edges are the original (triangulated) planar graph, and red vertices and red edges are the corresponding dual graph.

2.1 Properties of a dual graph

- There is an one-to-one correspondence between $F(G)$ and $V(G^*)$.
- There is an one-to-one correspondence between $E(G)$ and $E(G^*)$.

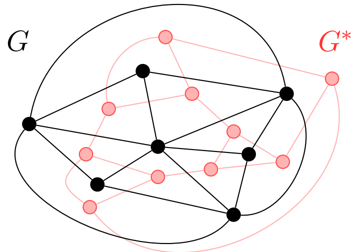

Figure 15.1: A dual of a planar graph

- There is an one-to-one correspondence between $V(G)$ and $F(G^*)$.
- G^* is a planar graph
- if G is triangulated, then G^* is 3-regular.

2.2 Interdigitating tree

Claim 15.6 (interdigitating trees)**.** *For any spanning tree $T \subseteq G$ of a planar graph G, the edges in G^* that do not correspond to edges in T form a spanning tree in G^**

Figure 15.2 below shows an interdigitating tree of a spanning tree of G. Black edges correspond to a spanning tree of G and red edges are the corresponding interdigitating tree of the spanning tree.

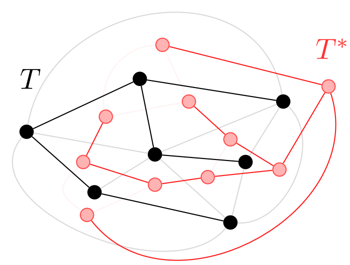

Figure 15.2: Interdigitating tree

Proof. Given a spanning tree $T \subseteq G$, let T^* be the graph formed by removing edges that correspond to edges in T. We want to show T^* is a spanning tree. It is sufficient to show that (a) T^* is acyclic and (b) T^* has $|V(G^*)| - 1$ edges.

First, we will show T^* is acyclic. Suppose for contradiction that T^* has a cycle C^*. Note that there is at least one face inside C^*, say u, and there is at least one face outside of C^*, say v. Since each face of G^* corresponds to a vertex of G, u and v are vertices in G. Furthermore, there is a path between u and v. This implies there exists an edge $e \in T$ on the path of u to v that crosses C^*. Let e^* be the corresponding edge of e. We have $e \in T$ and $e^* \in T^*$ which contradicts the construction of T^*.

We further want to show T^* is connected. In other words, we want to show T^* has $|V(G^*)|-1$ edges. Recall Euler's formula, $|V(H)| - |E(H)| + |F(H)| = 2$ for any planar graph H. By rearranging, we can obtain $|F(G)| - 1 = |E(G)| - |V(G)| + 1$. Furthermore, we know $|V(G^*)| = |F(G)|$.

$$
\begin{aligned}
|V(G^*)| - 1 = |F(G)| - 1 &= |E(G)| - |V(G)| + 1 \\
&= |E(G)| - (|V(G)| - 1) \\
&= |E(G)| - |E(T)| = |E(T^*)|
\end{aligned}
$$

Since T^* is acyclic and $|E(T^*)| = |V(G^*)| - 1$, T^* is a spanning tree of G^*. □

3 Proof of Lemma 15.3

Now we are ready to prove Lemma 15.3. We want to show the find a fundamental cycle C with $|\mathsf{In}(C)| \leq \frac{2n}{3}$ and $|\mathsf{Out}(C)| \leq \frac{2n}{3}$.

Proof. Given a spanning tree $T \subseteq G$, take the interdigitating tree T^* of T. Note every edge that is not in T has the corresponding edge in T^*. Thus there's a one-to-one correspondence between a fundamental cycle of G to an edge in T^*. We will choose an edge e^* in T^* instead of choosing a fundamental cycle of G.

We break the proof into two parts. We first pick an edge $e^* \in T^*$ such that resulting components in $T^* - e^*$ are 'balanced'. Then we claim that the fundamental cycle corresponding to the edge e^* is a separator of G.

Claim 15.7. *If T^* has max degree 3, then there exists an edge $e^* \in T^*$ such that each component of $T^* - e^*$ has at most $\frac{2n}{3}$ vertices.*

Proof. We will use the centroid of the tree to break the tree into a balanced portion.

Definition 15.8 (Centroid of a Tree)**.** A *centroid of a tree* T is a node $v \in T$ such that any tree in $T - v$ has size at most $\frac{n}{2}$.

We first find a centroid of T^* by following steps. Select an arbitrary node v of T^*. If v is a centroid we found a centroid. Otherwise, there exists a vertex u adjacent to v such that subtree rooted at u has size $> \frac{n}{2}$. Recursively search for a centroid with tree rooted at u.

Let v be the centroid of T^* we found. Note v has degree at most 3. Let T_1^*, T_2^*, T_3^* be the three connected components in the graph $T^* - v$. Without loss of generality assume $|T_1^*| \leq |T_2^*| \leq |T_3^*|$. This implies $|T_3^*| \geq \frac{n-1}{3}$ thus $|T_1^*| + |T_2^*| + |\{v\}| \leq \frac{2n}{3}$. Furthermore, since v is a centroid $|T_3^*| \leq \frac{n}{2}$. Let e^* be the edge between v and T_3^*, observe $T^* - e^*$ has components with size at most $\frac{2n}{3}$ vertices in G^*. □

We have shown that the cycle C corresponds to e^* separates vertices of G^* with size at most $\frac{2n}{3}$ each. This only separates faces of G. It remains to show C separates vertices of G into roughly equal parts. We will use a variant of Euler's formula to show that vertices of G are separated into roughly equal parts.

Recall Euler's formula, $V(H) - E(H) + F(H) = 2$ for any planar graph H. By rearranging we get $E(H) = V(H) + F(H) - 2$. Given a planar graph H, let H' be a triangulated graph of H. Then we have $3F(H') = 2E(H')$.

$$E(H') = V(H') + F(H') - 2$$
$$= V(H) + \frac{2}{3}E(H') - 2$$
$$\Rightarrow E(H) \leq E(H') = 3V(H) - 6 \tag{15.1}$$
$$\Rightarrow F(H) \leq F(H') = 2V(H) - 4 \Rightarrow V(H) \geq \frac{F(H)+4}{2} \tag{15.2}$$

We will use the above equations to show that C partitions vertices of G into roughly equal parts. Let D be the disk formed by C, i.e. $D := G[C \cup \mathsf{In}(C)]$. Then we get

$$\begin{aligned} V(D) &\geq \frac{F(D)+4}{2} && \text{(by the equation(15.2))} \\ &\geq \frac{\frac{1}{3}F(G)+4}{2} && \text{(by Claim 15.7)} \\ &= \frac{\frac{1}{3}(2V(G)-4)+4}{2} && \text{(by the equation (15.2), note } G \text{ is triangulated)} \\ &\geq \frac{1}{3}V(G) \end{aligned}$$

Thus we have $|\mathsf{Out}(C)| \leq \frac{2}{3}V(G)$. We can define $D' := G[C \cup \mathsf{Out}(C)]$ and apply the same arguments to get $|\mathsf{In}(C)| \leq \frac{2}{3}V(G)$.

□

4 Reduce to a Small Diameter Case

In previous sections, we found a separator of size at most $2\mathsf{diam}(G) + 1$. However, if G has a long diameter, then we do not get a separator of size $O(\sqrt{n})$. In this section, we will 'reduce' a planar graph G into a planar graph with a smaller diameter.

Given a planar graph G, we will first take the BFS tree rooted at an arbitrary vertex $r \in V(G)$. Let T be the resulting BFS tree. Let $V_\ell = \{v : d(v,r) = \ell\}$ be the set of vertices at layer ℓ. Let N be the number of layers in T. Observe that there is no edge between $V_{\ell-s}$ and $V_{\ell+2+t}$ for all $s, t \geq 0$. Thus V_ℓ separates $V_0 \cup \ldots \cup V_{\ell-1}$ and $V_{\ell+1} \cup \ldots \cup V_N$. Figure 15.3 summarizes notations for the BFS tree.

Pick the median i such that $|V_0| + \ldots = |V_{i-1}| < \frac{n}{2}$, and $|V_0| + \ldots + |V_i| \geq \frac{n}{2}$. If $|V_i| \leq \sqrt{n}$, then V_i is a separator with size $O(\sqrt{n})$.

Otherwise look at $\sqrt{n}$ layers right of V_i, namely $V_{i+1}, V_{i+2}, \ldots, V_{i+\sqrt{n}}$ (possibly $i + \sqrt{n} > N$, in which case we declare $V_j = \emptyset$ for all $j > N$). Note there exists $j \in [\sqrt{n}]$ such that $|V_{i+j}| \leq \sqrt{n}$. If no such j exists, then the size of V_{i+1} to $V_{i+\sqrt{n}}$ is more than the number of vertices, i.e. $\sum_{j=1}^{\sqrt{n}} |V_{i+j}| > \sqrt{n}\sqrt{n} = n$. Thus there must exist such j. Similarly we can find $k \in [\sqrt{n}]$ such that $|V_{i-k}| \leq \sqrt{n}$. Deleting $V_{i-k} \cup V_{i+j}$ splits G into three components. Let left, middle, and right be set of vertices in each component, i.e. $\mathsf{left} := V_0 \cup \ldots \cup V_{i-k-1}$, $\mathsf{middle} := V_{i-k+1} \cup \ldots \cup V_{i+j-1}$, and $\mathsf{right} := V_{i+j+1} \cup \ldots \cup V_N$. Figure 15.3 shows a visual representation defined sets.

Since i is the median, we know $|\mathsf{left}| < \frac{n}{2}$ and $|\mathsf{right}| < \frac{n}{2}$. If $|\mathsf{left}| \geq \frac{n}{3}$ or $|\mathsf{right}| \geq \frac{n}{3}$, then $V_{i-k} \cup V_{i+j}$ is a separator. Thus assume $|\mathsf{middle}| \geq \frac{n}{3}$.

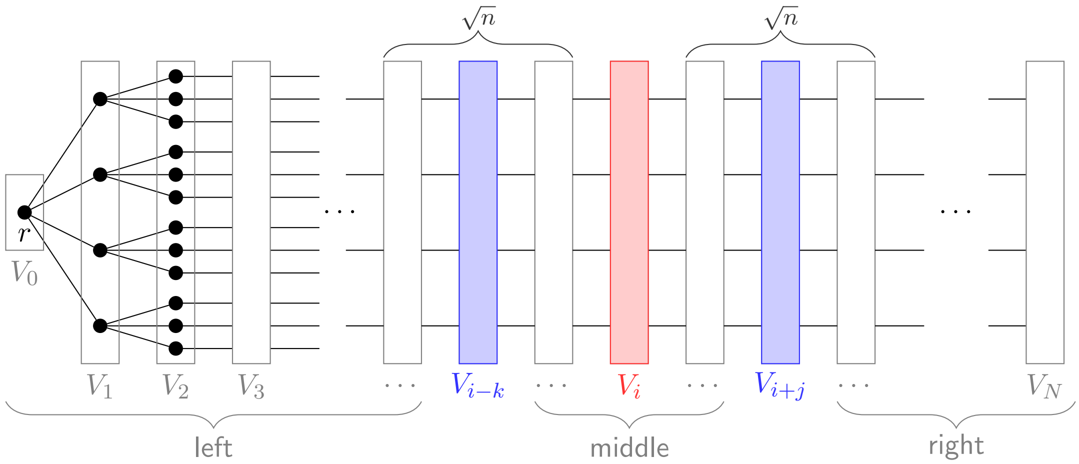

Figure 15.3: BFS of G

Let G' be the graph formed by $G[V_{i-k} \cup ... \cup V_{i+j} \cup \{r\}]$ with edges between each vertex in V_{i-k} and r as shown in Figure 15.4.

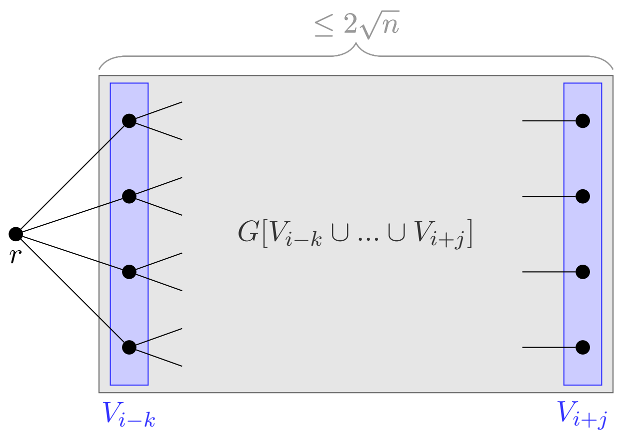

Figure 15.4: G'

Claim 15.9. $\mathsf{diam}(G') \leq O(\sqrt{n})$*, where* n *is* $|V(G)|$*.*

Proof. Consider any vertex v in G', note $d(v, r) \leq j + k + 1 \leq 2\sqrt{n} + 1$. Therefore, distance between any two nodes in G' is at most $4\sqrt{n} + 2 \leq O(\sqrt{n})$. □

Since the diameter of G' is $\leq O(\sqrt{n})$ we can obtain a separator of size $O(\sqrt{n})$ that separates G'. However, we only know $|V(G)| \geq \frac{n}{3}$, thus the separator creates components of size at least $\frac{n}{9}$.

Lemma 15.10. *Given an algorithm A that finds a separator of size $O(\sqrt{n})$ vertices such that removal of the separator leaves every connected component of size at most n/t vertices for $t = O(1)$, we can find a separator of size $O(\sqrt{n})$ whose removal leaves connected components of size at most $\frac{2n}{3}$.*

Proof. Applying the algorithm A once gives two components L and R with size at most n/t. We can recursively apply the algorithm to the bigger component ℓ times. Let union of all separators be our final separator. The resulting connected components will have size at most $(1/t)^\ell n$ with separator size $\ell O(\sqrt{n})$. We want $\frac{2n}{3} \leq (1/t)^\ell n$ implying $\ell \leq \log_{1/t}(2/3) = O(1)$. Thus separator size is also $O(\sqrt{n})$. □

We showed any planar graph G either has a good separator (V_i or $V_{i-k} \cup V_{i+j}$) or can be reduced to a graph G' with $\mathsf{diam}(G') = O(\sqrt{n})$. By Lemma 15.3, we can obtain a good separator of G' of size $O(\sqrt{n})$. The resulting separator together with $V_{i-k} \cup V_{i+j}$ form a separator that breaks G into $\leq \frac{8n}{9}$ vertices. Lastly, by Lemma 15.10, we can use above steps recursively to obtain a desirable separator that satisfy Theorem 15.1.

References

[LT79] Richard J. Lipton and Robert Endre Tarjan. A separator theorem for planar graphs. *SIAM J. Appl. Math.*, 36(2):177–189, 1979. 15.1