15-859FF: Coping with Intractability **CMU, Fall 2019**
HW #4 (out Wednesday Nov 6, 2019) **Due:** Wednesday November 20, 2019

You may discuss all problems on this HW with others, but groups of size at most 3, please. Please write down the names of your collaborators. Also, you should write your solutions yourself.

Exercises

Exercises are for fun and edification, please do not submit. **But do solve them, we may need ideas from there later in the course, or even in this HW!**

1. Exercise 7.6–7.9, and the first part of 7.10 (that trees have pathwidth $\lceil \log_2 n \rceil$).

2. Prove a weighted version of the planar separator theorem: given a planar graph G with vertex weights w_v, and total weight $W := \sum_v w_v$, there is a set of $O(\sqrt{n})$ vertices whose deletion breaks G into pieces each with weight at most $2W/3$.

3. An s-t series-parallel (SP) graph G is defined as follows: (a) it is an edge (s,t), or (b) it is obtained by taking a s_1-t_1 SP graph and a s_2-t_2 SP graph and identifying s_1 and s_2 into s and t_1 and t_2 into t, or (c) it is obtained by taking a s-t_1 SP graph and a s_2-t SP graph and identifying s_2 and t_1. A series-parallel (SP) graph G is one that is a s-t SP graph for some nodes s, t. Show that the treewidth of a series-parallel graph is at most 2, and this is best possible.

4. Prove Lemmas 7.2 and 7.4 in the Cygan book about constructing nice path/tree decompositions. This is also Exercises 7.1 and 7.2 in the book.

5. In lecture, we stated without proof that if a collection of intervals on the real line pairwise intersect, then they share a common endpoint. (This statement is proved in the lecture notes.)

   (a) Prove the following corresponding statement for trees: given a tree T, if a collection of (connected) subtrees $T_1, T_2, \ldots, T_\ell$ of T pairwise intersect (that is, $V(T_i) \cap V(T_j) \neq \varnothing$ for all $i, j \in [\ell]$), then they share a common vertex (that is, $V(T_1) \cap V(T_2) \cap \cdots \cap V(T_\ell) \neq \varnothing$).

   (b) Conclude that a graph of treewidth k does not contain K_{k+2} as a subgraph.

6. In class, we sketched out a proof that if a graph has treewidth k, then any minor of this graph has treewidth at most k. Flesh out this proof. Along with the previous exercise, conclude that a graph of treewidth k does not contain K_{k+2} as a *minor*.

7. For a finite set of points $P \subseteq R^d$, let $cg(P) := \frac{\sum_{p \in P}}{|P|}$ be the center of gravity. Let $\Delta(P, v) = \sum_{p \in P} \|p - v\|^2$. Show that

$$\Delta(P, v) = \Delta(P, cg(P)) + |P| \cdot \|cg(P) - v\|^2.$$

Problems

The Cygan et al book provides hints to its problems. Please try to solve these problems yourself before looking at these hints! Recall the notation $O^*(c^n)$ to denote $O(c^n \text{poly}(n))$; this helps avoid repeating polynomial factors in many places below. *You may state without proof the statements of any exercise above.* Please solve problem #5 and any three of the first four problems.

1. Cygan 7.18, do the max-cut and coloring.

2. Cygan 7.20

3. Show that a graph of treewidth k has pathwidth $O(k \log n)$. (Hint: one of the exercises helps.)

4. We mentioned Eppstein's theorem in lecture: that a planar graph with diameter D has treewidth at most $O(D)$. Prove a weaker version, that any planar graph G with diameter D has treewidth at most $O(D \log n)$. (Please be careful: if you decide to delete some set of vertices, the diameter of the graph may increase. You may use the weighted planar separator theorem without proof.)

5. In this problem, we show that the $t \times t$ grid $\boxplus_t$ has treewidth $\Omega(t)$. By considering the $\sqrt{n} \times \sqrt{n}$ grid, this shows the existence of planar graphs with treewidth $\Omega(\sqrt{n})$, which matches the $O(\sqrt{n})$ upper bound.

   Throughout the problem, fix a tree decomposition with tree T and bags $X_1, X_2, \ldots, X_\ell \subseteq V(\boxplus_t)$ as its nodes. The goal is to show that the tree decomposition has width $\Omega(t)$, that is, at least one bag X_j has size $\Omega(t)$.

   (a) For $i \in [t]$, define $V_i \subseteq V(\boxplus_t)$ to be the union of the i'th row and i'th column of $\boxplus_t$. Show that for all $i \in [t]$, the set of nodes X_j that contain at least one vertex in V_i (namely, $\{X_j \mid X_j \cap V_i \neq \varnothing\}$) forms a connected subtree of T.

   (b) Show that there exists a bag X_j containing at least one vertex from each V_i. (Hint: exercises.) Conclude that the width of the tree decomposition is $\Omega(t)$.