Planar graphs have treewidth $O(\sqrt{n})$.

Tight: $\sqrt{n} \times \sqrt{n}$ grid has treewidth $\sqrt{n}$.

Idea: recursive separators

1. Every vertex in $G$ belongs.
2. For each edge in $G$, there is a bag containing both of its endpoints.

Fix edge $(u,v)$.
Consider node $t_u$: highest node cont. $u$
$tv$: " " " " $v$.

Claim: either $t_u = tv$, or one is descendant of the other.

Pf: $sp$ not! $v$ lowest common ancestor
For each vertex \( v \), set of bags - ✓

4 width = \( O(\sqrt{n \log n}) \).

Grid minors and bidimensionality

"Certificates" for large treewidth?

1. Cliques: if \( G \) has \( K_t \) as a minor, then \( tw(G) \geq tw(K_t) = t-1 \).

"Converse"? Ideally: if \( tw(G) \geq f(t) \) then

G has \( K_t \) as a minor?

False: Define \( \mathbb{T}_t := t \times t \) grid.

\( tw(\mathbb{T}_t) = t \), but excludes \( K_5 \) as a minor (since planar).

2. Grids? If \( G \) has \( \mathbb{T}_t \) as a minor, then \( tw(G) \geq tw(\mathbb{T}_t) = t \).

"Converse"?

Polynomial Grid Minor Theorem:

[Chekuri, Chuzhoy '13 + improvements];

If \( tw(G) \geq \Omega(t^9 \text{ polylog } t) \), then \( G \) has \( \mathbb{T}_t \) as a minor,
and there's a polynomial \((n,t)\) time algo to find one.

Solve the FPT problems with Grid Minor Theorem.
Solving FPT problems with Grid Minor Theorem.

Min Vertex Cover

1. \( \text{min VC of } K_t \text{ is } \Omega(t^2) \).  
   \[
   \Rightarrow \text{ every VC size } \Omega(t^2)
   \]

2. If \( G \) has \( K_t \) as a minor, then \( \text{minVC}(G) \geq \Omega(t^2) \).

\[
\Rightarrow \text{ if } \text{tw}(G) \gg k^{4.5}, \text{ then } G \text{ has } K_{10,10k} \text{ as a minor}
\]
\[
\Rightarrow \text{minVC}(G) > k
\]

So if \( \text{minVC}(G) \leq k \), then \( \text{tw}(G) \leq \tilde{O}(k^{4.5}) \)

3. Compute tw-decomp of \( G \) of width \( \tilde{O}(k^{4.5}) \)
   - If cannot, then output NO.
   - Else, solve minVC on tw-decomp in time \( \tilde{O}(k^{4.5}) \).

Planar Grid Minor Theorem [Robertson, Seymour, Thomas]

If \( G \) is planar and \( \text{tw}(G) \geq 5t \), then \( G \) has \( K_t \) as a minor.
If $G$ is planar and $tw(G) = 5K$ then $G$ has $H_t$ as a minor.

If $tw(G) \geq 50K$, then $G$ has $H_{out}$ as a minor.

$\Rightarrow$ NO.

Algorithm: compute tw decomp of $G$ of width $O(5K)$.

If cannot, then output NO.

Else, solve min VC $2^{O(5K)}$.

Bidimensionality: $2^{O(5K)}$ time FPT algs on planar graphs:

1. Size of solution on $H_t$ is $L(t^2)$.
2. If $H$ minor of $G$ and $H$ has soln size $k$, then $G$ also has soln size $k$.
3. Given a tree decomp of width $t$, can solve problem in $2^{O(t)} n^{O(1)}$ time.

Ex: Longest path

3. $2^{O(5K \log K)}$ time

Ex: MaxIS: violate 2! Addly an edge can destroy IS. Can be fixed by

2. If $H$ is a contraction of $G$