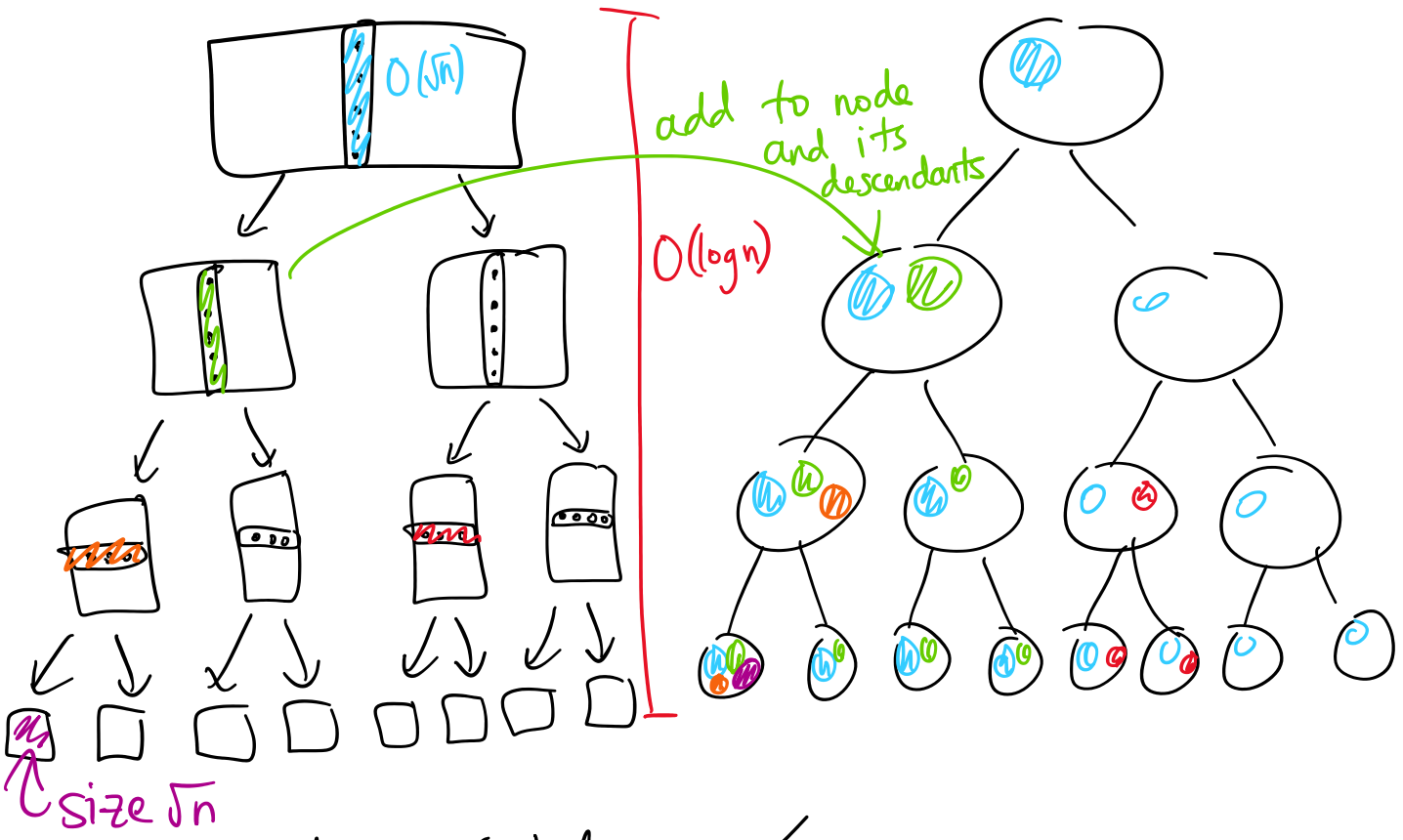
Planar graphs have treewidth $O(\sqrt{n})$

Tight: $\sqrt{n} \times \sqrt{n}$ grid has treewidth $\sqrt{n}$.

Idea: recursive separators



① Every vertex in G belongs... ✓

② For each edge in G, there is a bag containing both of its endpoints:

Fix edge (u,v).

Consider node t_u: highest node cont. u

t_v: " " " v.

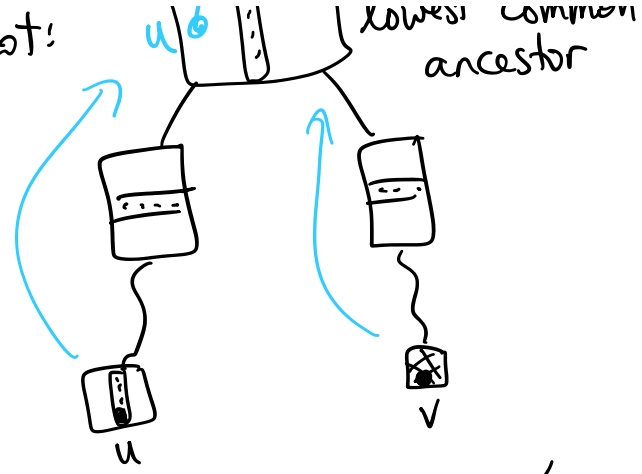
Claim: either $t_u = t_v$, or one is descendant of the other.

Contradiction.

Pf: sps not:

lowest common ancestor

Pf: sps not:



③ For each vertex v, set of bags — . ✓

④ width $= O(\sqrt{n}\log n)$.

Grid minors and bidimensionality

"Certificates" for large treewidth?

① Cliques: if G has K_t as a minor, then $tw(G) \geq tw(K_t) = t-1$.

"Converse"? Ideally: if $tw(G) \geq f(t)$ then G has K_t as a minor?

False: Define $\boxplus_t :=$ $t \times t$ grid.

$tw(\boxplus_t) = t$, but excludes K_5 as a minor (since planar)

② Grids? If G has $\boxplus_t$ as a minor, then $tw(G) \geq tw(\boxplus_t) = t$.

"Converse"?

Polynomial Grid Minor Theorem:

[Chekuri, Chuzhoy '13 + improvements]:

If $tw(G) \geq \Omega(t^9 \operatorname{polylog} t)$, then G has $\boxplus_t$ as a minor, and there's a polynomial (n,t) time algo to find one.

Solving FPT problems with Grid Minor Theorem.

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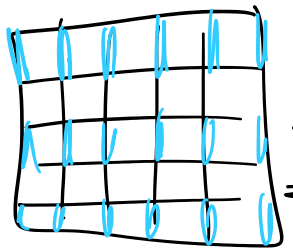
Min Vertex Cover

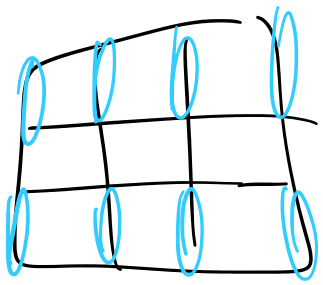
① min VC of $\boxplus_t$ is $\Omega(t^2)$.

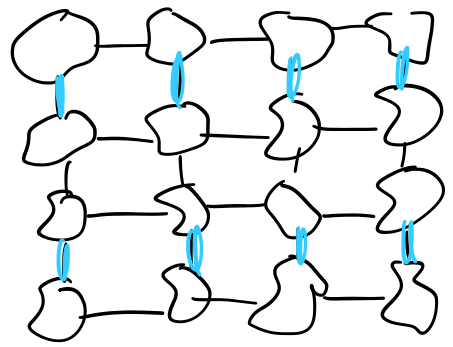


matching of size $\Omega(t^2)$

$\Rightarrow$ every VC size $\Omega(t^2)$

② If G has $\boxplus_t$ as a minor, then $\min VC(G) \geq \Omega(t^2)$.





$\Rightarrow$ If $tw(G) \gg k^{4.5}$, then

G has $\boxplus_{10\sqrt{k}}$ as a minor

$\Rightarrow \min VC(G) > k$

So if $\min VC(G) \leq k$, then $tw(G) \leq \tilde{O}(k^{4.5})$

③ Compute tw decomp of G of width $\tilde{O}(k^{4.5})$

If cannot, then output NO.

Else, solve minVC on tw decomp in time $2^{\tilde{O}(k^{4.5})}$.

Planar Grid minor Theorem [Robertson, Seymour, Thomas]

If G is planar and $tw(G) \geq 5t$ then

G has $\boxplus_t$ as a minor

If G is planar and $tw(G) \geq 5t$ then G has $\boxplus_t$ as a minor.

If $tw(G) \geq 50\sqrt{k}$, then G has $\boxplus_{10\sqrt{k}}$ as a minor $\Longrightarrow$ NO.

Algo: compute tw decomp of G of width $O(\sqrt{k})$.
If cannot, then output NO.
Else, solve min VC " " " " $2^{O(\sqrt{k})}$.

Bidimensionality: $2^{O(\sqrt{k})}$ time FPT algos on planar graphs:

① size of solution on $\boxplus_t$ is $\Omega(t^2)$

② If H minor of G and H has soln size k, then G also has soln size k.

③ Given a tree decomp of width t, can solve problem in $2^{O(t)} n^{O(1)}$ time

Ex: Longest path
③ $2^{O(\sqrt{k} \log k)}$ time

Ex: MaxIS: violate ② ! Adding an edge can destroy IS. Can be fixed by

②' If H is a <u>contraction</u> of G " " "