Lecture 4: Randomized Methods / Color Coding

Previous lecture: VC (via Kernelization, Bonded Search) d-regular set.
also reductions. Clique not in NL. \rightarrow Statement not in \text{WP}

today: Feedback Vertex Set
Longest path.

(FVS) multigraph \( G \), want to hit all cycles. Equiv. \( \text{XCSP} + G[V\setminus X] \text{ is linear} \)
\( (G[V] \subseteq G[V\setminus X] \text{ has no edge}) \).

Some reduction rules:
\( (G, k) \Rightarrow G \text{ has a } \text{FVS} \).
\( \begin{align*}
1. & \text{ if self-loop contain } v \Rightarrow (G - v, k - 1) \\
2. & \text{ drop degree 1 nodes } v \Rightarrow (G - v, k) \\
3. & \text{ if } G \Rightarrow \text{ has more than 2 parallel edges, drop all but 2.} \\
4. & \text{ if } v \text{ has degree 2, say } v \rightarrow w \Rightarrow \text{contraction of these edges.} \\
5. & \text{ if } k < 0 \Rightarrow \text{answer NO.}
\end{align*} \)

at the end, left with multigraph
\( G \) with min degree \( \geq 3. \)
\( \text{no self-loops, at most 2 parallel edges.} \)

Claim: at least \( \frac{1}{2} \) the edges of \( G \) have at least one endpoint in OPT.

Pf: Spes \( X \) in FVS of size \(< k \).
\( H = G[V\setminus X] \)

Want:
\( \# \text{edges incident on } \bar{X} \ CGPoint{\text{ of other edges}} \)
\( |\bar{J}| \geq |\bar{Y} \setminus X| \)

Let \( V_1, V_2, V_3 \) be partition of \( V \setminus X \) with degree (in \( H \) be \( k \))
\( \leq 1, \geq 2, \geq 3 \)
then since any degree in \( H \leq 2 \Rightarrow \leq 1|V_1| > |V_3| \geq 3 \)
but \( |\bar{J}| \geq 2|V_3| + 1, |V_2| \) (because degree \( \geq 3 \))
\( \geq |V_1| + |V_2| + |V_3| = |V\setminus X| \).
**Algorithm:** Recursive FVS ($G, k$)

- apply 5 rules to get ($G', k'$). Let $X_0$ be self-loop return.
- pick random edge, random endpoint.

$X_{rec} \leftarrow$ Recursive FVS ($G', k'$).

if $1 \leq k$ return NO. else return $X_{rec} \cup X_0$.

Success up $(\frac{1}{4})^{k-1}(\frac{1}{4}) \geq (\frac{1}{4})^k$.

$\Rightarrow$ time $= \Theta(k^3 \text{poly}(n))$.

Q: determinist (yes, later?)

Q: kernel size $O(k^3)$ later?

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**Longest Path**

- clearly in $	ext{XP}$.
- easy for bounded degree graphs.

1. color vertices with $k$ colors randomly.
   - the OPT path will be colored 1, 2, ..., $k$ (in mat order) up.
   - and we can find it in linear time.

$\Rightarrow$ time $= O(\frac{k^1 \text{poly}(n)}{k^k})$.

2. the OPT path will be rainbow-colored up
   - $\frac{k^1}{k^k} \gg \frac{\binom{k}{2} k^{1}}{k^k} = \frac{1}{k} k^k$.
   - And can find a rainbow path of length $k$ in time $2^k \cdot \text{poly}(n)$.

**Simple DP:** $T(S, v)$: does $\exists$ path with colored $S$ end at $v$?

- $T(S, w, u) = \max_{u \in S}$.
- $T(S, v) = \max_{u \in S} (T(S \setminus \{v\}, u))$.

So get $\left(2e\right)^{k^3 \cdot \text{poly}(n)}$.

Q: better? Yes, $4^k \cdot \text{poly}(n)$ using divide and conquer + color only.

Q: do? Later?

Q: improve? $2^k \cdot \text{poly}(n)$ by Williams [Koutis' paper works]?

Q: $16k$ by B. Lundt (undirected) yes, later?
Subgraph isomorphism: Given \( G, (n^4 + x^5) \), \( H \) (pattern, k-vertex)

- \( H \) is "simple" (path, tree, low tree-width)
  can do in time \( f(k) \cdot \text{poly}(n) \)
  degree depends on treewidth

- \( H \) is aliged to \( WJ \) hard.

What about bounded degree \( G \)? = max degree

- Can get an \( O(2^{\text{poly}(\log n)}) \) algorithm [AJW]

Here simple: \( 2^{dk} \cdot \text{poly}(n) \) algo.

- Color each edge red or blue. (red = "deleted" edges)
  - Make 3 copy of \( H \). then edges in \( \mathcal{W} \leq dk \). All red up \( \frac{1}{2} dk \)
  - Each copy of \( H \) them via \( H \) (w/2-dk prob)
  - So spend \( k! \cdot \text{time for each component. At most } O(n, k! \cdot \text{time}) \) for checking

- \( O(2^{dk} \cdot n, k!) \cdot \text{time} \)

Derandomization:

Idea: instead of considering all coloyages \( T \) \( \rightarrow \) \( \mathcal{D} \). Consider some "small" set

- Fix one of such \( T \) by \( \mathcal{D} \). Consider some "small" set

Folows st.

- Is a "good" function \( f : \mathcal{D} \rightarrow \mathcal{F} \) for whatever property we want.
  - Often, if success prob is \( p(k) \) then get \( |\mathcal{D}| = O(1/p(k) \cdot \log n) \).

\[ \left( m, k, \mathcal{D} \right) \text{ splitter is a family of functions from } \mathcal{D} \rightarrow \mathcal{F} \text{ if every set } S \text{ of size } k \text{ for } F \text{ s.t. } \# \text{elements in } S \text{ colored } c \text{ by } f \leq 1 \]

Thm: \( (m, k, \mathcal{D}) \) splitter of size \( \Omega \left( \frac{k \cdot \log k}{m} \right) \cdot \log n \) that can be constructed in

- \( O(n^4 + x^5) \cdot \text{time} \), \( O(k \cdot \log k) \cdot \text{time} \).

Observe: enumerate all these functions. One gives multicolored path. DP solves the rest.
(m,k) universal set is a family of subsets of [n] such that \( \forall S \subseteq [n], |S| \leq k \),

\[ S \text{ is shattered by } U \iff 2^{\binom{n}{k}} \leq 2^k \text{ has all } 2^k \text{ subsets.} \]

**Theorem:** Construction of universes of size \( \leq 2^k \) occurs within \( 2^{2^k} \) steps.

This can be used for the *random separation* algorithm.