Lecture 5

LP Duality*

In Lecture #3 we saw the Max-flow Min-cut Theorem which stated that the maximum flow from a source to a sink through a graph is always equal to the minimum capacity which needs to be removed from the edges of the graph to disconnect the source and the sink. This theorem gave us a method to prove that a given flow is optimal; simply exhibit a cut with the same value.

This theorem for flows and cuts in a graph is a specific instance of the **LP Duality** Theorem which relates the optimal values of LP problems. Just like the Max-flow Min-cut Theorem, the LP Duality Theorem can also be used to prove that a solution to an LP problem is optimal.

5.1 Primals and Duals

Consider the following LP

$$\begin{aligned} P = \max(2x_1 + 3x_2) \\ \text{s.t.} \quad 4x_1 + 8x_2 \le 12 \\ 2x_1 + x_2 \le 3 \\ 3x_1 + 2x_2 \le 4 \\ x_1, x_2 \ge 0 \end{aligned}$$

In an attempt to solve P we can produce upper bounds on its optimal value.

- Since $2x_1 + 3x_2 \le 4x_1 + 8x_2 \le 12$, we know $\text{OPT}(P) \le 12$.
- Since $2x_1 + 3x_2 \le \frac{1}{2}(4x_1 + 8x_2) \le 6$, we know $\text{OPT}(P) \le 6$.
- Since $2x_1 + 3x_2 \le \frac{1}{3}((4x_1 + 8x_2) + (2x_1 + x_2)) \le 5$, we know $\text{OPT}(P) \le 5$.

In each of these cases we take a positive linear combination of the constraints, looking for better and better bounds on the maximum possible value of $2x_1 + 3x_2$. We can formalize

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this, letting y_1, y_2, y_3 be the coefficients of our linear combination. Then we must have

$$\begin{aligned} 4y_1 + 2y_2 + 3y_2 &\geq 2 \\ 8y_1 + y_2 + 2y_3 &\geq 3 \\ y_1, y_2, y_3 &\geq 0 \\ \text{and we seek } \min(12y_1 + 3y_2 + 4y_3) \end{aligned}$$

This too is an LP! We refer to this LP as the dual and the original LP as the primal. The actual choice of which problem is the primal and which is the dual is not important since the dual of the dual is equal to the primal.

We designed the dual to serve as a method of constructing an upperbound on the optimal value of the primal, so if y is a feasible solution for the dual and x is a feasible solution for the primal, then $2x_1 + 3x_2 \leq 12y_1 + 3y_2 + 4y_3$. If we can find two feasible solutions that make these equal, then we know we have found the optimal values of these LP.

In this case the feasible solutions $x_1 = \frac{1}{2}, x_2 = \frac{5}{4}$ and $y_1 = \frac{5}{16}, y_2 = 0, y_3 = \frac{1}{4}$ give the same value 4.75, which therefore must be the optimal value.

5.1.1 Generalization

In general, the primal LP

$$P = \max(c^\top x \mid Ax \leq b, x \geq 0, x \in \mathbb{R}^n)$$

corresponds to the dual LP,

$$D = \min(b^\top y \mid A^\top y \geq c, y \geq 0, y \in \mathbb{R}^m)$$

where A is an $m \times n$ matrix.

When there are equality constraints or variables that may be negative, the primal LP

$$\begin{aligned} P = \max(c^\top x) & \\ \text{s.t.} \quad a_i x \leq b_i & \text{ for } i \in I_1 \\ a_i x = b_i & \text{ for } i \in I_2 \\ x_j \geq 0 & \text{ for } j \in J_1 \\ x_j \in \mathbb{R} & \text{ for } j \in J2 \end{aligned}$$

corresponds to the dual LP

$$\begin{aligned} D = \min(b^\top y) & \\ \text{s.t.} \quad y_i \geq 0 & \text{ for } i \in I_1 \\ y_i \in \mathbb{R} & \text{ for } i \in I_2 \\ A_j y \geq c_j & \text{ for } j \in J_1 \\ A_j y = c_j & \text{ for } j \in J_2 \end{aligned}$$

5.2 The Duality Theorem

The Duality Theorem will show that the optimal values of the primal and dual will be equal (if they are finite). First we will prove our earlier assertion that the optimal solution of a dual program gives a bound on the optimal value of the primal program.

Theorem 5.1 (The Weak Duality Theorem)**.** *Let* $P = \max(c^\top x \mid Ax \leq b, x \geq 0, x \in \mathbb{R}^n)$*, and let* D *be its dual LP,* $\min(b^\top y \mid A^\top y \geq c, y \geq 0, y \in \mathbb{R}^m)$*. If* x *is a feasible solution for* P *and* y *is a feasible solution for* D*, then* $c^\top x \leq b^\top y$*.*

Proof.

$$
\begin{aligned}
c^\top x &= x^\top c \\
&\leq x^\top (A^\top y) && \text{(Since } y \text{ feasible for } D \text{ and } x \geq 0) \\
&= (Ax)^\top y \\
&\leq b^\top y && \text{(Since } x \text{ is feasible for } P \text{ and } y \geq 0) \quad \square
\end{aligned}
$$

From this we can conclude that if P is unbounded ($\text{OPT}(P) = \infty$), then D is infeasible. Similarly, if D is unbounded ($\text{OPT}(D) = -\infty$), then P is infeasible.

Therefore we have the following table of possibilities for the feasibility of P and D.

$P \backslash D$	Unbounded	Infeasible	Feasible
Unbounded	no	yes	no
Infeasible	yes	???	???
Feasible	no	???	???

The Duality Theorem allows us to fill in the remaining four places in this table.

Theorem 5.2 (Duality Theorem for LPs)**.** *If* P *and* D *are a primal-dual pair of LPs, then one of these four cases occurs:*

1. *Both are infeasible.*
2. P *is unbounded and* D *is infeasible.*
3. D *is unbounded and* P *is infeasible.*
4. *Both are feasible and there exist optimal solutions* x, y *to* P *and* D *such that* $c^\top x = b^\top y$*.*

We have already seen cases 2 and 3 as simple consequences of the Weak Duality Theorem. The first case can easily be seen to occur: a simple example takes A to be a $\mathbf{0}$ matrix, b to be strictly negative, and c to be strictly positive). Therefore the only remaining case of interest is case 4.

Geometric Proof. Let P be the program $\max(c^\top x \mid Ax \leq b, x \in \mathbb{R}^n)$ and D be dual program $\min(b^\top y \mid A^\top y = c, y \geq 0)$.

Suppose x^* is an optimal feasible solution for P. Let $a_i^\top x \leq b_i$ for $i \in I$ be all the constraints tight at x^*. We claim that the objective function vector c is contained in the cone $K = \{x \mid x = \sum_{i \in I} \lambda_i a_i, \lambda_i \geq 0\}$ generated by the vectors $\{a_i\}_{i \in I}$.

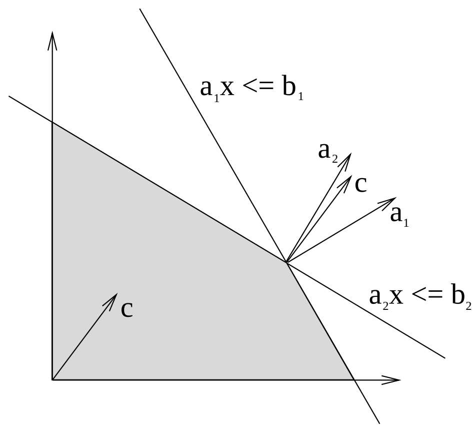

Figure 5.1: The objective vector lies in the cone spanned by the constraint vectors

Suppose for contradiction that c does not lie in this cone. Then there must exist a separating hyperplane between c and K: i.e., there exists a vector $d \in \mathbb{R}^n$ such that $a_i^\top d \leq 0$ for all $i \in I$, but $c^\top d > 0$. Now consider the point $z = x^* + \epsilon d$ for some tiny $\epsilon > 0$. Note the following:

- For small enough ϵ, the point z satisifes the constraints $Az \leq b$. Consider $a_j^\top z \leq b$ for $j \notin I$: since this constraint was not tight for x^*, we won't violate it if ϵ is small enough. And for $a_j^\top z \leq b$ with $j \in I$ we have $a_j^\top z = a_j^\top x^* + \epsilon a_j^\top d = b + \epsilon a_j^\top d \leq b$ since $\epsilon > 0$ and $a_j^\top d \leq 0$.
- The objective function value increases since $c^\top z = c^\top x^* + \epsilon c^\top d > c^\top x^*$.

This contradicts the fact that x^* was optimal.

Therefore the vector c lies within the cone made of the normals to the constraints, so c is a positive linear combination of these normals. Choose λ_i for $i \in I$ so that $c = \sum_{i \in I} \lambda_i a_i, \lambda \geq 0$ and set $\lambda_j = 0$ for $j \notin I$.

- We know $\lambda \geq 0$.
- $A^\top \lambda = \sum_{i \in [m]} \lambda_i a_i = \sum_{i \in I} \lambda_i a_i = c$.
- $b^\top \lambda = \sum_{i \in I} b_i \lambda_i = \sum_{i \in I} (a_i x^*) \lambda_i = \sum_{i \in I} \lambda_i a_i x^* = c^\top x^*$.

Therefore λ is a solution to the dual with $c^\top x^* = b^\top \lambda$, so by The Weak Duality Theorem, $\mathrm{OPT}(P) = \mathrm{OPT}(D)$. □

A somewhat more rigorous proof not relying on our geometric intuition that there should be a separating hyperplane between a cone and a vector not spanned by the cone relies on a lemma by Farkas that often comes in several forms. The forms we shall use are as follows

Theorem 5.3 (Farkas' Lemma (1894) - Form 1). *Given $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, exactly one of the following statements is true.*

1. *$\exists x \geq 0$ such that $Ax = b$.*
2. *$\exists y \in \mathbb{R}^m$ such that $y^\top A \geq 0$ and $y^\top b < 0$.*

Theorem 5.4 (Farkas' Lemma - Form 2)**.** *Given $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, exactly one of the following statements is true.*

1. *$\exists x \in \mathbb{R}^n$ such that $Ax \geq b$.*
2. *$\exists y \geq 0, y^\top A = 0$, and $y^\top b > 0$.*

Proofs. Left to the reader (homework 2). □

Duality Theorem using Farkas' Lemma. Let P be the program $\min(c^\top x \mid Ax \geq b, x \in \mathbb{R}^n)$ and D be dual program $\max(b^\top y \mid A^\top y = c, y \geq 0)$.

Suppose the the dual is feasible and its maximum value is δ. Let $P' = \{x \mid Ax \geq b, c^\top x \leq \delta\}$. If P' has a feasible solution, then P must also have a feasible solution with value at most δ. The LP P' is also equivalent to $\{x \mid Ax \geq b, -c^\top x \geq -\delta\}$.

Suppose for contradiction P' is infeasible. Then by Farkas' Lemma (Form 2) there exists $\begin{pmatrix} y \\ \lambda \end{pmatrix} \geq 0$ such that

$$(y^\top \ \lambda) \begin{pmatrix} A \\ -c^\top \end{pmatrix} = 0 \text{ and } (y^\top \ \lambda) \begin{pmatrix} b \\ -\delta \end{pmatrix} > 0$$

This implies $y^\top A - \lambda c^\top = 0$ and $y^\top b - \lambda\delta > 0$.

- If $\lambda = 0$, then $y^\top A = 0$ and $y^\top b > 0$. Choose $z \geq 0$ such that $A^\top z = c$ and $b^\top z = \delta$. Then for $\epsilon > 0$,

$$\begin{aligned} A^\top(z + \epsilon y) &= 0 \\ z + \epsilon y &\geq 0 && \text{(Since } y \geq 0) \\ b^\top(z + \epsilon y) &= \delta + \epsilon b^\top y \\ &> \delta \end{aligned}$$

  so $z + \epsilon y$ is a feasile solution of D with value greater than δ, a contradiction.

- Otherwise we can scale y and λ to make $\lambda = 1$ (since $y, \lambda \geq 0$), so $y^\top A = c^\top$ and $y^\top b > \delta$. This means y is a solution to D with value greater than δ, a contradiction.

Therefore P' is feasible, so P is feasible with value at most δ. By The Weak Duality Theorem, $\text{OPT}(P) = \delta = \text{OPT}(D)$. □

In the next couple of lectures, we will continue to explore duality, and its applications.