

Lecture 4

Avis-Kaluzny and the Simplex Method*

Last time, we discussed some applications of Linear Programming, such as Max-Flow, Matching, and Vertex-Cover. The broad range of applications to Linear Programming means that we require efficient algorithms to solve it.

4.1 The Avis-Kaluzny Algorithm

The Avis-Kaluzny Algorithm is designed to find a basic feasible solution (BFS) of a given set of constraints. Its input: $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ such that

$$\begin{aligned} Ax &\leq b \\ x &\geq 0 \end{aligned}$$

Note that this algorithm requires the nonnegativity constraint to be present. The algorithm proceeds as follows:

4.1.1 The Algorithm

Terminology The set of equations obtained in step 1 is called the tableau or dictionary. The set basic variables B is a set of variables, initially a set of slack variables, which has the property that each variable is alone on one side of each equation.

All other variables are called non-basic variables.

A basic variable is said to "exit the basis" when step 4 causes it to no longer be a basic variable, and a non-basic variable is said to "enter the basis" when step 4 causes it to become a basic variable.

Starting the algorithm:

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Step 1: Slack Introduce slack variables $x_{n+1}\dots x_{n+m}$ for each of the inequalities in \mathcal{A} , turning them into equations. Obviously, but importantly, like the other variables, these slack variables also must be nonnegative.

Step 2: Try it Set all non-basic variables to 0, and see if this results in a BFS (does this cause the basic variables to be positive?) If it is, output the value of each variable under this assignment (or just the non-slack variables).

Step 3: B-rule Take the basic variable of minimum index that is negative in the assignment in step 2; call it x_b . On the other side of the selected equation, if all coefficients are negative, return this equation as evidence that the equation is INFEASIBLE. Otherwise, select the non-basic variable of minimum index that has a positive coefficient; call it x_n .

Step 4: Pivot Solve the selected equation for x_n in Step 3. This means that x_n enters the basis, and x_b exits the basis. For every instance of x_n in other equations, substitute the other side of the selected equation. Go to step 2.

4.1.2 Example

Suppose we were given the following set of equations:

$$\begin{aligned}x_1 + 2x_2 &\leq 6 \\2x_1 + x_2 &\leq 6 \\2x_1 + 2x_2 &\geq 7 \\x_1, x_2 &\geq 0\end{aligned}$$

After performing step 1, our tableau is:

$$\begin{aligned}x_3 &= -x_1 - 2x_2 + 6 \\x_4 &= -2x_1 - x_2 + 6 \\x_5 &= 2x_1 + 2x_2 - 7\end{aligned}$$

We perform step 2: Set non-basic variables to 0. In this case, x_1 and x_2 are our non-basic variables. This means that

$$\begin{aligned}x_3 &= 6 \\x_4 &= 6 \\x_5 &= -7\end{aligned}$$

This is not feasible as $x_5 < 0$. x_5 is the lowest-indexed variable below 0, so we proceed to step 3. We select the equation $x_5 = 2x_1 + 2x_2 - 7$ and note x_1 has a positive coefficient, so

we select it for the pivot step.

We perform step 4 and solve for x_1 : $x_1 = \frac{1}{2}x_5 - x_2 + \frac{7}{2}$. Go to step 2. Our tableau is currently:

$$\begin{aligned}x_3 &= -\frac{1}{2}x_5 - x_2 + \frac{5}{2} \\x_4 &= -x_5 + x_2 - 1 \\x_1 &= \frac{1}{2}x_5 - x_2 + \frac{7}{2}\end{aligned}$$

We set non-basic variables to 0.

$$\begin{aligned}x_3 &= \frac{5}{2} \\x_4 &= -1 \\x_1 &= \frac{7}{2} \\x_1, x_2, x_3, x_4, x_5 &\geq 0\end{aligned}$$

This is not a basic feasible solution. We continue, selecting x_2 to pivot about x_4 :

$$\begin{aligned}x_3 &= -\frac{3}{2}x_5 - x_4 + \frac{3}{2} \\x_2 &= x_5 + x_4 + 1 \\x_1 &= -\frac{1}{2}x_5 - x_4 + \frac{5}{2}\end{aligned}$$

With the basic variables set to 0, we obtain

$$\begin{aligned}x_3 &= \frac{3}{2} \\x_2 &= 1 \\x_1 &= \frac{5}{2}\end{aligned}$$

Thus the solution $x_1 = \frac{5}{2}, x_2 = 1, x_3 = \frac{3}{2}, x_4 = 0, x_5 = 0$ works and we return $x_1 = \frac{5}{2}, x_2 = 1$. It can be verified that this is a solution of the original problem.

4.1.3 Correctness

Step 2 At each stage, a satisfying assignment to the tableau also satisfies the original tableau, as each tableau is a linear combination of previous tableaus.

Step 3 Suppose that you have an equation where the basic variable is negative when all non-basic variables are 0, and the coefficient of all non-basic variables is negative. This means that if the non-basic variables in the equation are non-negative, the basic variable is a sum of nonpositive terms, at least one of which is negative, and thus it is impossible to make the basic variable positive, thus making the entire system infeasible.

The most important thing to verify about this algorithm is that it actually terminates.

Theorem 4.1. *The Avis-Kaluzny algorithm terminates.*

Proof. First of all, note that there are only finitely many possible tableaus for any given set of constraints, as a set of basic variables uniquely determines a tableau given a set of initial constraints.

So if this algorithm does not terminate, there must be a cycle among the possible tableaus.

Suppose that we have a cycle in the set of possible tableaus. We will assume that x_{n+m} enters and leaves the basis during this cycle.

Justification Suppose that x_{n+m} does not enter and leave the basis during this cycle. Then it follows that the tableau formed by removing x_{n+m} in all equations (remove the equation containing x_{n+m} if x_{n+m} is a basic variable, and remove all instances of x_{n+m} from all equations otherwise) will also cycle, and we consider that tableau instead.

When x_{n+m} enters the basis, there must be an equation of the form

$$x_b = k + \sum_{a \in \{1..m+n\}} c_a x_a + c_{m+n} x_{m+n}$$

where c_i for all $i \in \{1..m+n\}$ is the coefficient of x_i in this equation, k is a constant, $c_i \leq 0$ for all $i < m+n$, $k < 0$, and x_b is the basis variable.

$k < 0$ because x_b must be negative when the non-basic variables are 0, and x_i for all $i < m+n$ must be nonpositive as otherwise they would be chosen over x_{m+n} to be the pivot.

This means that every solution to these equations with $x_1 \dots x_{n+m-1} \geq 0$ has $x_{n+m} > 0$; otherwise x_b would be a sum of nonpositive values with at least one negative term.

When x_{n+m} leaves, if all non-basic variables are set to 0, as x_{n+m} is lexicographically last, all basic variables must be nonnegative on this assignment in order for x_{n+m} to be chosen. This is an assignment such that $x_1 \dots x_{n+m-1} \geq 0$ but $x_{n+m} < 0$, contradicting the fact we proved when x_{n+m} entered the basis!

By contradiction, there is no cycle in the set of possible tableaus, and therefore the Avis-Kaluzny algorithm terminates.

□

4.2 The Simplex Algorithm

The Simplex algorithm is an algorithm that steps from a BFS to a better BFS, improving the quantity that we are trying to maximize (or minimize) at every step of the way.

Its input: $A \in \mathbb{R}^{m \times n}$, $b, c \in \mathbb{R}^m$ such that

$$\begin{aligned} \max \quad & c^\top x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

4.2.1 The algorithm

Step 1: Slack Introduce slack variables $x_{n+1} \dots x_{n+m}$ for each of the inequalities in \mathcal{A} , turning them into equations. These equations become your "extended" tableau, which is simply a tableau with the maximizing condition added on. You assume that you already have a BFS in the sense that you assume that if all non-basic variables are set to 0, then the basic variables will be non-negative.

Step 2: Improve In the expression to maximize, select a variable with a positive coefficient. This is called the improving variable. If this does not exist, then simply set all basic variables to

Step 3: Pivot Pivot on the equation that limits the improving variable the most (sets the lowest upper bound on it) by solving for the improving variable in that equation and then substituting for that variable in all other equations, including the equation for the quantity to be improved. Note that the upper bounds are conservative and assume that other variables are 0. If there is no limit on the improving variable, there is no limit to the quantity that needs to be maximized either.

4.2.2 Intuition

This algorithm can be viewed as "crawling along the edges" of the polytope representing the set of feasible solutions. In particular, this figure represents the crawl that we perform below.

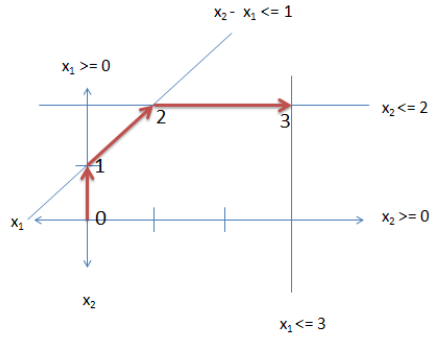


Figure 4.1: The traversal through the polytope in the Simplex example

4.2.3 Example

Consider the following equation:

$$\begin{aligned}
 \max \quad & z = x_1 + x_2 \\
 \text{s.t} \quad & -x_1 + x_2 \leq 1 \\
 & x_2 \leq 2 \\
 & x_1 \leq 3 \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

In step 1, we would make the following extended tableau:

$$\begin{aligned}
 \max \quad & z = x_1 + x_2 \\
 \text{s.t} \quad & x_3 = 1 + x_1 - x_2 \\
 & x_4 = 3 - x_1 \\
 & x_5 = 2 - x_2
 \end{aligned}$$

Note this state represents the BFS 0 in Figure 4.1. as when the non-basic variables are set to 0 $x_1 = x_2 = 0$.

In step 2, suppose that we choose x_2 as our improving variable. Note that $x_3 = 1 + x_1 - x_2$ limits x_2 to 1 as we assume x_1 is 0 for these purposes (we do not want to introduce an equation that causes a negative basic variable when non-basic variables are set to 0). $x_5 = 2 - x_2$ limits x_2 to 2 but this is not as strict of a bound. $x_4 = 3 - x_1$ does not relate to x_2 .

In step 3, we pivot, so the tablau now looks like this:

$$\begin{aligned}
\max \quad & z = 1 + 2x_1 - x_3 \\
\text{s.t} \quad & x_2 = 1 + x_1 - x_3 \\
& x_4 = 3 - x_1 \\
& x_5 = 1 - x_1 + x_3
\end{aligned}$$

Note this represents the BFS 1 in Figure 4.1, as when the non-basic variables are set to 0 $x_1 = 0$ and $x_2 = 1$.

Now, we must choose x_1 as our improving variable. x_1 is not limited by the x_2 equation, but is limited to 3 by the x_4 equation and 1 by the x_5 equation. So we pivot on the x_5 equation to obtain the tableau

$$\begin{aligned}
\max \quad & z = 3 - 2x_5 + x_3 \\
\text{s.t} \quad & x_2 = 2 - x_5 \\
& x_4 = 2 - x_3 + x_5 \\
& x_1 = 1 - x_5 + x_3
\end{aligned}$$

Note this represents the BFS 2 in Figure 4.1, as when the non-basic variables are set to 0 $x_1 = 1$ and $x_2 = 2$.

We choose x_3 as the improving variable. It is limited to 2 by the x_4 but is not limited by the x_1 equation.

So we pivot on the x_4 equation:

$$\begin{aligned}
\max \quad & z = 5 - x_5 - x_4 \\
\text{s.t} \quad & x_2 = 2 - x_5 \\
& x_3 = 2 - x_4 + x_5 \\
& x_1 = 3 - x_4
\end{aligned}$$

Note this represents the BFS 3 in Figure 4.1, as when the non-basic variables are set to 0 $x_1 = 3$ and $x_2 = 2$. Notice this point is optimal.

There is no improving variable anymore, so we set $x_5 = 0$, $x_4 = 0$ and thus $x_3 = 2$, $x_2 = 2$, $x_1 = 3$, and so we return $x_1 = 3$ and $x_2 = 2$ as the optimal solution.

4.2.4 Issues

Unbounded? If this is the case, there will at some point be an improving variable that has no constraints on it.

No improvement? If you get an upper bound of 0 on a variable, your pivot will not increase the quality of the solution by making the non-basic variables 0. Simply continue on in this case.

It is possible for the simplex algorithm to stall! (see the homework)

Also, there are a few choices that you have to make in this algorithm. In step 2, there are often several improving variables to choose. In step 3, how do you resolve ties between two equally constraining equations?

These choices actually affect some key runtime properties of the Simplex algorithm.

Pivoting Rule(choice of improving variable)	Can cycle?	Can take exponential time?
Largest Coefficient	Yes	Yes
Largest Increase	Yes	Yes
Steepest-Edge(most parralel to obj. vector)	Yes	Yes
Lexicographical Rules (Bland's rules)	No!	Yes
Least-Entered	Yes	Yes
Clairvoyant (we know best route to opt.)	Yes	?

Hirsch's Conjecture stated that the edge-vertex graph of any polytope with n points in d -dimensional space has diameter at most $n-d$, which implies that a clairvoyant Simplex algorithm could reach the optimum in linear time. This was disproven in 2010 by Francisco Santos Leal, but it is still possible (and an open problem) whether or not the diameter of a polytope is polynomial in the number of points.