Linear Programming and Semidefinite Programming **CMU 15-859E, Fall 2011**

HOMEWORK 5

Due: Tuesday, November 15.

Ground rules: *same as for Homework 1.*

Remarks: We say that two relaxations for the same problem are *equivalent* if they have the same optimum value on all instances. Typically one shows this by giving an effective method for transforming a solution for one into a solution for the other which is at least as good.

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1. Canonical IP relaxation. The canonical LP relaxation for instance $\mathcal{C}$ of a CSP is the relaxation of a natural integer program (IP) which exactly captures the optimization problem for $\mathcal{C}$. Write that IP. Give a brief "interpretation", explaining why it is exact.

2. Equivalence I.

(a) Show that the canonical LP relaxation for Max-Cut is equivalent to the simple LP relaxation for Max-Cut given in Lecture 10.

(b) Show that the canonical LP for Max-kSat is equivalent to the simpler LP relaxation you (presumably) gave in Problem 5b of Homework 2. *(Remark: for Max-Sat, the canonical LP relaxation is not of polynomial size, but the equivalent one from Homework 2 is.)*

3. XXX. Consider the canonical SDP relaxation for instance $\mathcal{C}$ of a CSP, with the "consistent first moments" ① and the "consistent second moments constraint" ②. You may assume (really, this is without loss of generality) that $\mathcal{C}$ is "connected", meaning that the hypergraph of its variables and constraint-scopes is connected. In this problem you will show that we get an equivalent SDP relaxation if we drop the consistent first moments constraint.

(a) Show that in any solution to the SDP with just ②, we have the property that the random variable $J_v := I_v[0] + \cdots + I_v[q-1]$ is the *same* for all $v \in V$. (Call that same random variable J.)

(b) Consider the version of the SDP with just ② using using vectors and dot-products, rather than pseudoindicator random variables. What does part (a) tell us about the vectors? (Write $z_\emptyset \in \mathbb{R}^N$ for the vector corresponding to J.)

(c) Explain how a vector solution can be transformed to an equal-value one with the extra property that "$z_\emptyset$" is the vector $(1, 1, \ldots, 1) \in \mathbb{R}^N$.

(d) Show that this transformed vector solution satisfies SDP condition ①.

4. Goemans–Williamson, analyzed properly. Let $A\colon [-1,1] \to [0,1]$ be the function $A(\sigma) = \frac{1}{\pi}\cos^{-1}(\sigma)$. Let $\alpha\colon [-1,1] \to [0,1]$ be the "lower convex envelope" of A. (Formally: the pointwise-supremum of all convex functions upper-bounded by A; informally: the "largest" convex function whose graph fits underneath-or-equal-to the graph of A.)

(a) Sketch a plot of A and α. Give a short formula for $\alpha(\sigma)$; it should refer to two certain numbers, one roughly .879 and one roughly $-.689$. You need not justify your work here and may refer informally to these numbers.

(b) (Bonus.) Show that $-.689$ is actually $\cos(\theta)$, where $\theta \in (0,\pi)$ is the solution to $\tan(\theta/2) = \theta$. What is .879?

(c) Show that the Goemans–Williamson algorithm is in fact an $(\alpha(1-2\beta), \beta)$-approximation algorithm for Max-Cut for each $\beta \in [\frac{1}{2}, 1]$.

(d) You know that $\cos(x) = 1 - \frac{x^2}{2} \pm O(x^4)$, right? (And that $\cos(\pi + x) = -\cos(x)$, etc.?) Show that the Goemans–Williamson algorithm is a $(1 - (2/\pi)\sqrt{\epsilon} - o(\sqrt{\epsilon}), 1-\epsilon)$-approximation algorithm for small $\epsilon > 0$.

5. Random threshold. Consider the "Order$_{10}$" CSP mentioned briefly in Lecture 13: the variables take values in the set $[10]$ and the constraints are all of the form "$x_i < x_j$". Give a polynomial-time $(1 - O(\epsilon), 1-\epsilon)$-approximation algorithm for this CSP. (Hint: use the canonical LP relaxation. The trick is to design and analyze the right rounding scheme.)

6. Dual of the Day. Remember the SDP defining the Lovász ϑ-function from Lecture 11:

$$\begin{aligned} &\min t \\ \text{s.t. } &\langle v_i, v_j \rangle = t \quad i,j \in V, i \nsim j, i \neq j \\ &\langle v_i, v_i \rangle = 1 \quad \forall i \in V \end{aligned}$$

The optimal value of this SDP was $\frac{1}{1-\vartheta(G)}$, which is non-positive for all non-clique graphs.

(a) Associating the vertices with $\{1, \ldots, n\}$, show that the following SDP has value $\frac{1}{\vartheta(G)-1}$:

$$\begin{aligned} \max\ & X \bullet (e_{n+1}e_{n+1}^\top) \\ \text{s.t. } & X \bullet (e_i e_j^\top + e_{n+1}e_{n+1}^\top) = 0 \quad i,j \in [n], i \nsim j, i \neq j \\ & X \bullet (e_i e_i^\top) = 1 \quad \forall i \in [n] \\ & X \succeq 0 \end{aligned} \tag{1}$$

where X is an $(n+1) \times (n+1)$ matrix, and $e_1, e_2, \ldots, e_{n+1}$ are elementary unit vectors.

(b) Using the recipe from Lecture 12, write down the dual of this SDP. Recall, it says that given *symmetric* matrices A_j, C, and scalars b_j, the dual of the SDP

$$\max\{C \bullet X \mid A_i \bullet X = b_i, X \succeq 0\}$$

is

$$\min\{\textstyle\sum_j b_j y_j \mid \sum_j y_j A_j \succeq C\}.$$

(c) Infer that the dual you derived is equivalent to the following SDP:

$$\begin{aligned} \min \textstyle\sum_i Z_{ii} & \\ \text{s.t. } Z_{ij} = 0 & \qquad \forall i,j \in V, i \sim j \\ \textstyle\sum_{i \neq j} Z_{ij} \geq 1 & \\ Z \succeq 0 & \end{aligned} \tag{2}$$

where Z is an $n \times n$ symmetric matrix.

(d) Rearrange the above SDP to show that the following SDP has value $\vartheta(G)$:

$$\begin{aligned} \max \textstyle\sum_{i,j \in V} B_{ij} & \\ \text{s.t. } B_{ij} = 0 & \qquad \forall i,j \in V, i \sim j \\ \textstyle\sum_{i \in V} B_{ii} = 1 & \\ B \succeq 0 & \end{aligned} \tag{3}$$