Hardness of Facility Location

Know that $1 \cdot \text{vs} \cdot (1-\frac{1}{e})$ problem for Max Coverage is \text{NP-hard}

(i.e. distinguishing instances where $k$ sets cover all of $U$

$\text{vs. } k \text{ sets, cover at most } (1-\frac{1}{e}+\epsilon) \text{ of } U$

is \text{NP hard}).

Will use this to show that $1 \cdot \text{vs} \cdot * (1.463)$ problem for Facility Location

is hard.

i.e. we will give a map of the form

\[
\begin{array}{c}
\text{Max Coverage} \\
1-\frac{1}{e} + \epsilon
\end{array}
\]

\[
\begin{array}{c}
\text{FacLoc} \\
\text{maxFK}(1.463)
\end{array}
\]

Show this approx is hard as well.

Take instance $(U, S = \{S_1, S_2 \ldots S_m\})$ s.t. $US_i = U$.

Construct the fac loc instance

\[
\begin{array}{c|c|c}
L & R & 1 & 2 & \cdots & n \\
\hline
1 & 0 & 0 & 0 & \cdots & 0 \\
2 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
n & 0 & 0 & 0 & \cdots & 0
\end{array}
\]

\begin{itemize}
\item facility costs : \begin{align*}
\forall i \in L & : f_i = \infty \\
\forall i \in R & : f_i = f
\end{align*}
\item clients = $L$.
\item distances : if element $j \in \text{Set } S_i$$ \Rightarrow d(i,j) = 1$
\item else $d(i,j) = 3$
\end{itemize}

\begin{itemize}
\item element set
\item = clients = "potential facility locations"
\end{itemize}

YES instance \begin{itemize}
\item where $k$ sets cover all elements.
\end{itemize}

$\Rightarrow k$ facilities cover clients at dist 1.

\begin{align*}
cost &= k.f + n.1
\end{align*}

NO instance: any $nk$ sets cover at most $n(1-\frac{1}{e})$ elements.
\[ \Rightarrow \text{in this case } \text{cost} = f(\alpha k) + n(1-1/e^\alpha) + \frac{n^2}{e^\alpha} \]
\[ = f(\alpha k) + (1 + \frac{2}{e^\alpha}) n \]

\[ \Rightarrow \text{best solution } = \min_{\alpha} \left\{ f(\alpha k) + (1 + \frac{2}{e^\alpha}) n \right\} \]

Taking derivatives, get that \( \alpha^* = \ln\left(\frac{2n}{f k^2}\right) \) minimizes this quantity.

\[ \Rightarrow \text{regardless of what facilities chosen, } \text{cost} \geq f(\alpha k) + (1 + \frac{2}{e^\alpha}) n \]

Now: our hardness becomes

\[ (k f + n) - \text{ vs } -(k f \ln\left(\frac{2n}{f k^2}\right) + n + k f) \]

Again, suppose define \( x = \frac{k f}{n} \), this becomes

\[ n(1+x) - \text{ vs } -n\left(x \ln(\frac{2n}{f k^2}) + 1 + x\right) \]

Can set \( f \) to maximize this ratio \( \frac{x \ln(\frac{2n}{f k^2}) + 1 + x}{1 + x} = 1 + \frac{x \ln(\frac{2n}{f k^2})}{1 + x} \)

Wolfram Alpha says this is \( 1.463 \).

This is the current best hardness result for Facility Location.

[Falk Kruelle '98].