Submodularity

$f: 2^{\mathcal{U}} \to \mathbb{R}$ f submodular if $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$. $\forall A, B \subseteq \mathcal{U}$

$\iff \forall A \subseteq B \subseteq \mathcal{U}, i \notin B$ $\quad f_A(i) \geq f_B(i)$.

$f(A \cup i) - f(A)$

Monotone vs Non-monotone.

Examples:
- linear
- Budget additive $\min(\sum a_i x_i, B)$.
- rank function of matroids
- set coverage functions.
- cut functions — in graphs & hypergraphs
  — directed vs undirected.
- Entropy of a set of random vars

- $f+g$ is submod
- αf, $\alpha \geq 0$.
-

Q: maximizing a submodular function
minimizing a submodular functions. —

convex or concave?
- easy to minimize ⇒ kinda like convex
- but 1-dimension is concave (decreasing marginals).

$$P_f = \{ x \in \mathbb{R}^{\mathcal{U}} \mid x(S) \leq f(S) \ \forall S,\ x \geq 0 \} \qquad EP_f = \{ x \in \mathbb{R}^{\mathcal{U}} \mid x(S) \leq f(S) \ \forall S \}$$

We'll assume that $f(\emptyset) = 0$. (if $f(\emptyset) < 0$ then EP_f is empty
$f(\emptyset) > 0$ then can set $f(\emptyset) = 0$ and still submodular)

~~Algo: Greedy~~: want to solve $\boxed{\max\ w^T x \ \text{ st } \ x \in EP_f}$

- Sort $e_1, e_2, \ldots, e_n$ st $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_n)$ $\qquad A_i = \{e_1, \ldots e_i\}$.
- $x_i \leftarrow f(A_i) - f(A_{i-1})$
- Exercise: show this is optimal (fit a dual).

Optimizing monotone f over P_f has same thing;

Submodular Function Minimization

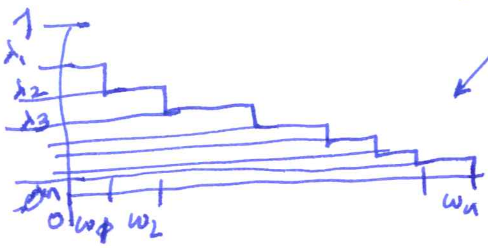
For any function $f: \{0,1\}^n \to \mathbb{R}$, def. $f^{Lovasz} = f^L : [0,1]^n \to \mathbb{R}$ as.

$$f^L(w) = \mathop{\mathbb{E}}_{\lambda \in_R [0,1]} \left[f(\{ i \mid w_i \geq \lambda \}) \right].$$

[Lovasz Extension].

$$= \sum_{i=0}^{n} \lambda_i f(S_i)$$

where $\sum_0^n \lambda_i = 1$, $\lambda_i \geq 0$, $S_0 = \emptyset \subseteq S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n$.



- **Fact:** if f submodular, $f^L(w) = \max\{ w^T x \mid x \in EP_f \}$.

  Pf: just run the greedy algo for ↑. Will get the above defⁿ.

- **Corollary:** f^L is convex when f is submodular (in fact, iff). won't prove it here

  Pf: By lemma $f^L(w) = w^T x^*$ for some $x^* \in EP_f$

  so if $w = \lambda a + (1-\lambda) b \Rightarrow f^L(w) = \lambda w^T a + (1-\lambda) w^T b$
  $\leq \lambda f^L(a) + (1-\lambda) f^L(b)$. ✓

- **Thm:** $\min_{S \subseteq U} f(S) = \min_{w \in [0,1]^n} f^L(w)$.

  Pf: if S^* is min on the left, set $w = \mathbb{1}_{S^*}$ gives $\cancel{f^L(\mathbb{1}_{S^*}) \leq}$ $\min_w f^L(w) \leq f(S^*)$.

  For other direction, say w^* is min on RHS, then
  $f^L(w^*) = \mathbb{E}[\text{several sets}]$ and one will be at least as good.

⇒ can optimize over the Lovasz extension to get submodular minimization.

∃ other algos to do this: - strongly poly algos, and combinatorial algos too.

Not today.

Combinatorial Algorithms for Submod Max (Approximately).

Max a submodular function subject to a cardinality constraint:- Say monotone $f(\emptyset)=0$.

Greedy: pick $\underset{e:\, S+e \text{ is feasible}}{\operatorname{argmax}} f_S(e)$. until done.

Thm: [Nemhauser Wolsey Fischer '78] $(1-1/e)$ apx for monotone submod f, cardinality constraint.

Pf: Say $f(C) = OPT$.

Cheap 2-apx.

Claim that at any point S_i, adding all elts in C will give value $f_{S_i}(C)$

$\Rightarrow \exists$ elt a_i in C s.t. adding it will give value $\frac{f_{S_i}(C)}{K} \geq \frac{f_{S_K}(C)}{K}$

submod. $\Rightarrow$ subadditivity   submodularity.

$\Rightarrow$ each time get value $\frac{f_{S_K}(C)}{K} \times K. \Rightarrow f_{S_K}(C)$.

$\Rightarrow$ final value $f(S_K) \geq f_{S_K}(C) \quad \Rightarrow f(S_K) \geq f(S_K \cup C) - f(S_K)$

$\Rightarrow 2f(S_K) \geq f(S_K \cup C) \Rightarrow f(S_K) \geq \frac{1}{2} f(S_K \cup C)$

$\geq \frac{1}{2} f(C)$.

monotone.

Better: $f_{S_i}(\underbrace{a_i}_{\text{good elt}}) \geq \frac{1}{K}(f(C) - f(S_i))$

$$
\begin{aligned}
f(C) - f(S_{i+1}) &= f(C) - f(S_i) - f_{S_i}(a_i) \\
&\leq \left(1-\frac{1}{K}\right)(f(C) - f(S_i)) \\
&\leq \left(1-\frac{1}{K}\right)^{i+1} \cdot f(C)
\end{aligned}
$$

$$\Rightarrow f(C) - f(S_K) \leq \left(1-\frac{1}{K}\right)^K f(C) \leq \frac{1}{e} \cdot f(C)$$

$$\Rightarrow f(S_K) \geq (1-1/e) f(C). \quad \blacksquare$$

This is optimal unless P = NP [Feige].

Submod. max (matroid constraint)

Thm: [Fisher Nem. Wolsey '78] $\frac{1}{2}$ apx. for monotone ~~constraints~~ submod.

S greedy, C = optimal by strong exchange ppty, $\exists$ PM on $S \Delta C$ st $S - a_i + b_i \in \mathcal{I}$.

Let $S_i = \{a_1 \ldots a_i\}$

since we could have chosen b_{i+1} instead of a_{i+1} we know

$$f_{S_i}(a_{i+1}) \geq f_{S_i}(b_{i+1})$$

$$\Rightarrow f(S) = \sum_{i=0}^{r-1} f_{S_i}(a_{i+1}) \geq \sum_i f_{S_i}(b_{i+1}) \geq \sum_i f_{S}(b_{i+1})$$

$$= f_S(C) = f(S \cup C) - f(S)$$

$$\Rightarrow f(S) \geq \frac{1}{2} f(S \cup C) \underset{\text{mono.}}{\geq} \frac{1}{2} f(C)$$

Non-monotone opt ~~[Double Greedy]~~ [Unconstrained]

$f(\emptyset) = 0$, f non-negative.

- Random set $S \subseteq U$ $\quad E f[S] \geq \frac{1}{4}$ OPT !
- Local search Start with ~~elt~~ A max value
  do single swaps until local opt.
  Return set or its complement. S or $U \setminus S$.

} $\frac{1}{3}$-apx.

Pf: Claim: if S is local opt ~~and~~ then for any $I \subseteq S$ or $I \supseteq S$, $f(I) \leq f(S)$.

Pf: Else could add/drop elt to improve.

if $I \subseteq S$ then $\exists a \in S \setminus I$ st $f(S-a) > f(S)$.

~~Just consider $S, S-a_1, S-a_1-a_2, \ldots I$~~

~~one step increases value Say $S-a_1 \ldots -a_i$, to $S-a_1 \ldots -a_i$~~

~~$\Rightarrow$ adding a_i to ... decreases val~~

sps $f(I) > f(S)$, $I \subseteq S$ ~~$\Rightarrow$ adding a_i to~~

$\Rightarrow$ on walk from I to S $\geq$ one elt caused value to drop. say x

so $f(I \cup J \cup x) < f(I \cup J)$

$\Rightarrow f((S-x)+x) < f(S-x)$ ☺

$\Rightarrow f(S \cap C) \leq f(S)$, $f(S \cup C) \leq f(S)$

Now $\underbrace{\overset{A}{f(S\cup C)}}_{\le f(S)} + \overset{B}{f(U\setminus S)} \ge \overset{A\cup B}{f(U)} + f(C\setminus S) \ge f(C\setminus S)$.

$\Rightarrow 2f(S) + f(U\setminus S) \ge f(C\setminus S) + f(C\cap S) \underset{\text{subadditivity}}{\ge} f(C)$.

(since $f(S) \ge f(C\cap S)$)

$\Rightarrow \min(f(S), f(U\setminus S)) \ge \frac{1}{3} f(C)$. ∎

Multilinear Relaxation of Submodular Functions

For maximization problems, the multilinear extension is the method of choice.

$$F(\bar{x}) = \mathop{\mathbb{E}}_{\hat{x}:\, i\text{th bit } 1 \text{ w.p. } x_i}\left[f(\hat{x})\right] = \sum_S \Big(\prod_{i\in S} x_i \prod_{j\notin S}(1-x_j)\Big) f(S)$$

Claim: given any downwards closed polytope P (i.e. $x\in P$, $y\le x$ coordinatewise $\Rightarrow y\in P$) (that we can optimize over) and monotone submodular fn f, can maximize $F(x): x\in P$ $(1-1/e)$-approximately in polytime. $F(x_{final}) \ge (1-1/e)OPT$

Double Greedy for monotone maximization :-

[Homework] for analysis

start with $A_0=\emptyset$, $B_0=U$.

for $i=1$ to n.

$\alpha = f(A_{i-1}+e_i) - f(A_{i-1})$

$\beta = f(B_{i-1}\setminus e_i) - f(B_{i-1})$. // Lemma: $\alpha+\beta \ge 0$.

if either α or $\beta < 0$ do the obvious thing

else w.p. $\frac{\alpha}{\alpha+\beta}$ $A_i \leftarrow A_{i-1}+e_i$ (but e_i still in B_{i-1} : $B_i \leftarrow B_{i-1}$)

w.p. $\beta/\alpha+\beta$ $B_i \leftarrow B_{i-1}\setminus e_i$ (~~not~~ in A_{i-1} : $A_i \leftarrow A_{i-1}$).

at the end: $A_n = B_n$. // in fact: $A_i = B_i \cap \{1..i\}$.

Submodular Functions in Graphs / Proofs

~~Halls Thm~~ **Edmonds:** G digraph with "root" r. Then $\exists$ k edge-disjoint arborescences in G

iff $\forall$ set $\underbrace{S \subseteq V-r}_{S \text{ "interesting"}}$, $\delta^{out}(S) \geq k$.

Pf: One direction is immediate.

For other dir, let's construct F spanning arb st $\delta^{out}_{G-F}(S) \geq k-1$ $\forall S$ interesting, then done.

Start building F from root. Suppose have some arb F st $V(F) \neq F$. and $\delta^{out}_{G\setminus F}(S) \geq k-1$ $\forall S$ int.

Will find F' with one more edge.

- Call set S critical if $\delta^{out}_{G\setminus F}(S) = k-1$.

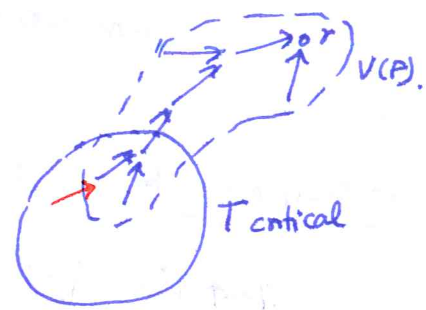
  ↑ interesting   Clearly $S \cap V(F)$.

Submod!

Claim: ~~Intersection~~/union of critical sets is critical.

Pf: $(k-1)+(k-1) = \delta^{out}(A) + \delta^{out}(B) \geq \delta^{out}(A \cup B) + \delta^{out}(A \cap B)$
$\geq k-1+k-1.$ ■

- Let T be a minimal critical set $\not\subseteq V(F)$ [if no such set $T = V$]



$\exists$ $(u,v) \in E$ st $u \in$ ~~V(F)~~ $T \setminus V(F)$
$v \in T \cap V(F)$

else $\delta^{out}_{G}(T \setminus V(F)) = \delta^{out}_{G\setminus F}(T \setminus V(F))$

$= k-1$ by criticality. (this contradicts assumption)

Claim: (u,v) cannot leave a critical set T'. Else $T \cap T'$ would be critical, contradicting minimality of T..

$\Rightarrow$ $T+(u,v)$ is a good arborescence, and we go on.