SDPs :

HW 5/6

A $n \times n$ symmetric matrix A is psd if :- $(A \succeq 0)$

[Löwner ordering $A \succeq B \overset{def}{\iff} A - B \succeq 0$.]

(a) $x^T A x \geq 0 \quad \forall x$.

(b) All evs of A are nonnegative

(c) $A = PDP^T$ where $D = \text{diag}(\lambda_1, \lambda_2 \cdots \lambda_n)$
$= \sum_i \lambda_i v_i v_i^T$ cols of P are eigenvectors v_i of A.

define $A^{1/2} = PD^{1/2}P^T$ (as before) and hence $A = A^{1/2} A^{1/2}$.

(d) there exist jointly distributed random variables $X_1, X_2 .. X_n$ such that
$A_{ij} = E(X_i X_j) \quad \forall ij$

(e) there exist vectors $v_1, .. v_n$ such that
$A_{ij} = \langle v_i, v_j \rangle$

(f) A can be written as BB^T for some matrix B (just take rows of $B = v_i$)

~~SDP a semidefinite program~~

Fact: given a matrix A, can check if it is psd in (strongly) polynomial time.
In fact if A is not psd, can return x st $x^T A x < 0$.

OK: what is a semidefinite program?

(i) It's a linear program where the variables are indexed by $i,j \in [N]$ (wlog)
say $(x_{ij})_{ij \in N}$

And we have an extra constraint $X = (x_{ij}) \succeq 0$ (psd)

(ii) It's a linear program where instead of linear constraints on variables (and linear obj fn on vars) we have constraints / obj on inner products of vector variables $\sum_{ij} a_{ij}^{\ell} \langle v_i, v_j \rangle \geq b_i \quad \forall \ell \in 1..m$.

By definition (e) these are equivalent.

Thm: Using the ellipsoid algorithm, we can solve SDPs in time $\text{poly}(\text{size of instance}, \log \frac{1}{\varepsilon})$ to within $(1+\varepsilon)$
(assuming bounds on the solution size !! ← fix)

Why is this useful? Two examples.

① Eigenvalues: given A (symmetric) suppose we want to find its largest eigenvalue λ_1.

then SDPs we say

$$\boxed{\begin{array}{ll} \max & A\bullet X \\ \text{st} & X\bullet I = 1 \\ & X \succeq 0 \end{array}} \quad \text{or} \quad \boxed{\begin{array}{ll} \min & t \\ \text{st} & tI - A \succeq 0. \end{array}}$$

Here $A\bullet B = \sum_{ij} a_{ij} b_{ij} = tr(A^T B)$.

is called the ~~Schur~~ Frobenius inner product product.

nb. Schur-Hadamard takes A, B gives $C = A\circ B$ ↑ open circle $C_{ij} = A_{ij}B_{ij}$

this is same as $\{\max \sum a_{ij} x_{ij} \mid \sum x_{ii} = 1,\ X \succeq 0\}$

sps $X = x_i x_i^T$ (rank 1)

then to max $A\bullet X$ $(= A\bullet x_i x_i^T = x_i^T A x_i)$ we'd want to take $x_i = v_1$ to get λ_1

Now ~~the max is obtained~~

for general $X = \sum \mu_i x_i x_i^T$ might as well put all mass on one μ_i, with $x_i = v_i$

~~Two natural ways to write SDPs~~: (later).

② Max-Cut: $x_v \in \{0,1\}$, $y_e \in \{0,1\}$ $\quad \max \sum y_e$ st y_e & x are "consistant"

want $y_e \le |x_u - x_v|$

~~OK for 0-1 solution.~~
~~But can set~~

$$y_e \le x_u + x_v$$
$$y_e \le 2 - (x_u + x_v)$$

OK for 0/1 soln
for $x_u = 1/2$
set $y_e = 1\ \forall e$.

∃ graphs where cut value ≈ ½$|E|$ but the LP has value $|E|$.

⇒ "Integrality gap" of 2.

So the LP is weak!

Can make it stronger by adding in constraints $\sum_{e\in C} y_e \le |C| - 1$ for all odd cycles C.

But still can construct graphs where LP $\simeq (2-\varepsilon)$ OPTimum Max Cut.

[BTW: how to solve the LP? Ellipsoid! Exercise on optional HW #7]

[In fact NO LP can get better than 2, unless has ~~exponential~~ super poly size] [CLRS 12]

OK: sdps to the rescue. Let's change the basis to $x_v \in \{-1,1\}$. and write

OPT as: $$\max_{x_i \in \{\pm 1\}} \sum_{ij \in E}' \frac{\|x_i - x_j\|^2}{4} = \sum_{ij \in E}' \frac{\|x_i\|^2 + \|x_j\|^2 - 2\langle x_i, x_j\rangle}{4}$$

unit vectors

$$= \sum_{ij \in E}' \frac{1 - \langle x_i, x_j\rangle}{2}$$

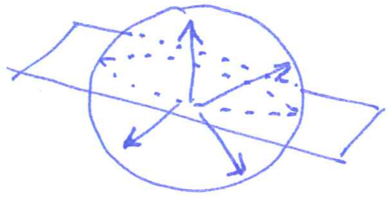
"Relax" this to get $$\boxed{\max_{\|v_i\|_2 = 1} \sum_{ij \in E}' \frac{1 - \langle v_i, v_j\rangle}{2}}$$

Thm: [GW] this SDP has an integrality gap of at most $\alpha_{GW} = 0.87856\ldots$

[ ] ——— at least ✓

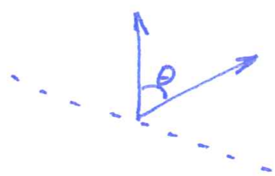
[KKMO] Assuming the unique games conjecture, Max cut does not have any algos better than $0.878\ldots$

Pf: Claim: $\Pr[ij$ cut by a random hyperplane$]$
$\geq \alpha \cdot SDP_{ij} \leftarrow \frac{1 - \langle v_i, v_j\rangle}{2}$



Pf: let v_i, v_j be unit vectors with angle θ

then $SDP_{ij} = \frac{1 - \cos\theta}{2}$



$\Pr[ij \text{ cut}] = \frac{\theta}{\pi}$

$$\Rightarrow \alpha = \min_{0 \leq \theta \leq \pi} \frac{\theta}{\pi} \cdot \frac{2}{1 - \cos\theta} = 0.878\ldots$$

∎

Here's another way to get this SDP:–

we would like to solve $\max \sum_{ij}' \left(\frac{1 - y_{ij}}{2} \right)$ st $y_{ij} = x_i x_j$ ← quadratic!
$x_i \in \{+1, -1\} \ \forall i$

So cannot solve that. However, can throw down "valid" inequalities (linear)

that are consistent with $y_{ij} = x_i x_j$:— $\quad \forall c, \; c^T Y c = c^T x x^T c = (c^T x)^2 \geq 0.$

(aka $Y = xx^T$)

$\Rightarrow$ are throwing in the (infinitely many) constraints $c^T Y c \geq 0 \quad \forall c \in \mathbb{R}^n$.

use Ellipsoid to separate for these (need strong separation oracle, etc.).

Good: seen what SDPs are, how they are useful. More powerful than LPs.

Next up: · SDP duality

· a couple more examples. → ~~Max Cut~~ Indep. set & coloring? → SOS? → Grothendieck?

·

SDP Duality:

Two canonical ways of writing LPs:

$$\min c^T x \quad Ax = b,\; x \geq 0 \quad \Longleftrightarrow \quad \max b^T y \quad \text{st } A^T y \leq c \quad \text{(duals)}$$

~~max~~ ~~Ax~~ ~~x≥0~~ $\Longleftrightarrow$ $\min c^T x$, $\bar{a}_i^T \bar{x} = b_i$, $x \geq 0$ and $\max b^T y$ st $\sum_i y_i \bar{a}_i \leq \bar{c}$.

$A = \begin{pmatrix} -\,a_1^T\,- \\ -\,a_i^T\,- \\ -\,a_m^T\,- \end{pmatrix}$

Here are two canonical SDP formulations.

Let $A_1, A_2 \ldots A_m$ be symmetric matrices. then also C

$$\boxed{\begin{aligned} &\min C \bullet X \\ &\text{st } A_i \bullet X = b_i \quad \forall i \in [m] \\ &X \succeq 0 \end{aligned}} \quad \text{and} \quad \boxed{\begin{aligned} &\max \sum_{i=1}^m b_i y_i \\ &\sum A_i y \preceq C. \end{aligned}}$$

In fact these are duals of each other: [soon.] [~~max cut~~ Lagrangian duals]

an example :— $\min\; -A \bullet X$ st $-I \bullet X = -1$, $X \succeq 0$ $\Rightarrow$ $\max\; -y$ st $-Iy \preceq -A$ or $\min\; y$ st $yI - A \succeq 0$

which we saw before.

Why are these duals of each other? See Boyd & Vandenberghe, e.g. but let's ~~at least show~~ weak duality:-

Thm: if $X \succeq 0$ and $\bar{y}$ are feasible ~~duals~~ solutions for the primal and dual pair, then

$(P) \geq (D)$

Pf: We need the following fact:-

{Fact: if A is symmetric then $A \bullet X \geq 0$ for all psd X $\Leftrightarrow$ A is psd.}

Now: Let's write $P = \left\{ \begin{array}{l} \min\ C \bullet X \\ A_i \bullet X = b_i \\ X \succeq 0 \end{array} \right\}$ and $D = \left\{ \begin{array}{l} \max \sum b_i y_i \\ \text{st} \sum A_i y_i + S = C \\ S \succeq 0 \end{array} \right\}$

slack.

then $(P) = C \bullet X = \left(\sum_{i=1}^{m} A_i y_i + S \right) \bullet X$

$= \sum_{i=1}^{m} y_i (A_i \bullet X) + S \bullet X = \sum b_i y_i + \underbrace{\geq 0}_{\text{by Fact.}}$

$\geq (D)$

∎

Aside: dual cones. $\mathcal{C} \subseteq \mathbb{R}^d$ be a cone :- ~~contain~~ closed under positive linear combs.

then $\mathcal{C}^* \subseteq \mathbb{R}^d$ is the polar cone

$\mathcal{C}^* = \{ x \in \mathbb{R}^d \mid \langle x, y \rangle \geq 0 \quad \forall y \in \mathcal{C} \}$

Fact: $\mathcal{C}^* = \mathbb{R}^d$, $\mathcal{C}^* = \{0\}$

$\mathcal{C} = \mathbb{R}^d_{\geq 0}$ then $\mathcal{C}^* = \mathbb{R}^d_{\geq 0}$ } self dual

$\mathcal{C} = \text{psd}$ $\quad \mathcal{C}^* = \text{psd}$

$\min\ c^T x$ | $\max\ b^T y$

$a_i x = b_i \quad \forall i$ $\Rightarrow$ dual is $\sum_i a_i y_i + s = c$

$x \in \mathcal{C}$ | $s \in \mathcal{C}^*$

and same proof goes through. (cone duality).

∎

But alas, no strong duality: there are examples where primal and dual are both feasible but $P^* \neq D^*$.

However, under certain "regularity" conditions strong duality holds. E.g. if primal has a strictly feasible sol^n (i.e. $X \succ 0$ positive definite) then dual opt is attained and primal opt = dual opt.

See BV04 and others for details on other sufficient conditions.

Now: a couple more examples —

(1) Coloring 3-colorable graphs. [~~Kar~~ Lovasz, Alon Kahale, Karger Motwani Sudan]

G be 3 colorable. Then can color it by $O(\sqrt{n})$ colors [Wigderson].

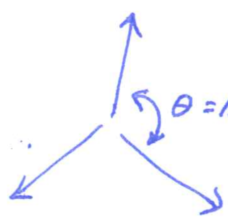
In fact, if we ~~con~~ repeatedly pick vertices of degree $\geq \Delta$ and color it and its nbrhood by 3 colors, we use $3(n/\Delta) +$ # colors to color graph with max degree Δ

$$\leq 3(n/\Delta) + \underset{\text{Brooks}}{(\Delta+1)} \leq O(\sqrt{n}).$$

Do better? Avrim: $O(n^{3/8})$ hard: 3 vs 4.

KMS: write an SDP ~~o~~ (feasibility)

$$\begin{cases} \langle v_i \cdot v_j \rangle = -\frac{1}{2} & \forall i \sim j \\ \|v_i\|^2 = 1 & \forall i \end{cases}$$



Feasible for 3-colorable graphs.

Now: (round 1) Pick K random (thru origin) hyperplanes. These divide up ball into 2^K parts.

Look at any vertex. $\Pr[v \text{ shares part with } \text{~~a~~ some nbr}] \leq \Delta \cdot \Pr[v, w \text{ in same part, if } u \sim v]$

$$= \Delta \cdot (1/3)^K \leq \frac{1}{2} \text{ if } K = \log_3(2\Delta)$$

$\Rightarrow$ If we throw away all nodes that share a part with a nbr, have at least $n/2$ vertices in expectation. Each part is an IS.

$\Rightarrow$ use 2^K colors, color $1/2$ vertices. Repeat $\log n$ times.

$\Rightarrow E[\#\text{colors in all}] = \underset{\uparrow \text{ rounds}}{2^k} \cdot \log n = 2^{\log_3 2\Delta} \cdot \log n = O(\Delta^{\log_3 2} \cdot \log n)$

$= \tilde{O}(\Delta^{0.631\ldots})$

$\Rightarrow$ By W's trick : $\frac{n}{\Delta} + \Delta^{0.631} \Rightarrow$ gives $n^{0.36\ldots}$ not better than Arrim's $n^{0.375}\ldots$

OK : harder work.

Thm: Find an IS of size $\Omega\left(\frac{n}{\Delta^{1/3}}\right) \Rightarrow$ gives a coloring ~~f~~ with $\Delta^{1/3}\log n$ colors

$\Rightarrow$ using W's trick $\Rightarrow n^{1/4}$ $\boxed{\text{much better!!}}$

Pf: Pick a random ~~vector~~ gaussian $g \in \mathbb{R}^n$

and take all vertices i st $\langle g \cdot v_i \rangle \geq t$ for a threshold t.

- Recall: $\langle g \cdot v_i \rangle \sim N(0, \|v_i\|_2^2) = N(0,1)$.

  g: multivariate gaussian; v_i: unit vector

$\Rightarrow \Pr[\langle g, v_i \rangle \geq t] \leq \mathcal{N}(t) \leftarrow$ upper tail of gaussian.

$\Rightarrow E[\text{Cap} \cap V] = n \cdot \mathcal{N}(t)$.

- Claim: conditioned on $i \in$ Cap, a nbr $j \sim i$ is in the Cap w/p $\lessapprox \mathcal{N}(t)^3$.

Pf: angle is $120°$. So if use rotational symmetry, $v_i = e_i$

$v_j = -\frac{1}{2}e_i + \frac{\sqrt{3}}{2}e_j$

$\Rightarrow \Pr[\langle g \cdot v_j \rangle \geq t \mid \langle g \cdot v_i \rangle \geq t]$

$= \Pr\left[-\frac{g_1}{2} + \frac{g_2 \cdot \sqrt{3}}{2} \geq t \mid g_1 \geq t\right]$

$\leq \Pr[g_2 \geq \sqrt{3}t] \approx \mathcal{N}(t)^3$. ∎

$\Rightarrow$ Set t st $\mathcal{N}(t)^3 \approx \frac{1}{2\Delta} \Rightarrow \Pr[i \text{ has no nbrs in cap} \mid i \text{ in cap}] \geq \frac{1}{2}$

$\Rightarrow E[\text{size of IS in Cap}] \geq \frac{1}{2} n \mathcal{N}(t) = \Omega\left(\frac{n}{\Delta^{1/3}}\right)$.

∎

<u>SoS</u>: A polynomial (multivariable) (n-vars.) is <u>positive</u> if (also called psd)

$P(\bar{x}) \geq 0 \quad \forall\, x \in \mathbb{R}^n$.

: A poly is SoS if $\exists$ polys $h_1, h_2 \ldots h_t$ st $P(\bar{x}) = \sum_{i=1}^{t} (h_i(x))^2$.

Clearly P is SOS $\Rightarrow$ P is positive. ← any univariate poly is sum of 2 squares!

- <u>Opposite is not true</u>. (for $n \geq 2$)

(<u>though</u>: P is positive iff P = ratio of SOS polynomials). [Emil Artin].
(sums of squares of rational fns) Hilbert's 17th problem

<u>Q</u>: Can we check if P is an SOS polynomial? Let $P(x_1, x_2 \ldots x_n)$ of degree d.
Yes, via SDPs.

$P = \sum' c_\alpha \bar{x}^\alpha$.
$\alpha \in \mathbb{Z}^n_{\geq 0}$
$\|\alpha\|_1 \leq d$

Suppose $P = h(x)^2$

where $h(x) = \sum' a_\alpha x^\alpha$

$\Rightarrow h(x)^2 = \sum'_{\alpha,\beta} a_\alpha a_\beta \cdot x^{\alpha+\beta}$

$= (aa^T) \bullet X$

psd matrix. $\quad X_{\alpha,\beta} = x^{\alpha+\beta}$.

<u>So ask</u>: does $\exists\, Q$ st (of size $\binom{n}{\leq d} \times \binom{n}{\leq d}$)

$Q \bullet \text{Test}_\gamma = c_\gamma \quad \forall\, \gamma$

has 1's in locations α, β st $\alpha + \beta = \gamma$

$Q \succeq 0$.

has size $\leq n^{O(d)}$.

see example next page

> Similarly if $P = \sum h_i(x)^2$
>
> $\Rightarrow P = \underbrace{\sum' (a_i a_i^T)}_{\text{etc.}} \bullet X$
>
> want this to be Q.

<u>N.b.</u> h_i must have degree $\leq d$ else cannot cancel the high degree terms.

<u>SoS proof systems</u>:

Motzkin's form: $\underbrace{x^4y^2 + y^4x^2 + z^6}_{\geq 3\sqrt[3]{x^6y^6z^6}\ \text{(AM-GM)}} - 3x^2y^2z^2$ is psd

univariate poly

<u>but not SoS</u>.

Example:

Is $p(x) = 2x_1^4 + 2x_1^3 x_2 - x_1^2 x_2^2 + 5x_2^4$. positive? SOS?

univariate so the two are identical

$$p(x) = \begin{pmatrix} x_1^2 \\ x_2^2 \\ x_1 x_2 \end{pmatrix}^T \underbrace{\begin{pmatrix} 2 & -\alpha & 1 \\ -\alpha & 5 & 0 \\ 1 & 0 & -1+2\alpha \end{pmatrix}}_{Q:} \begin{pmatrix} x_1^2 \\ x_2^2 \\ x_1 x_2 \end{pmatrix}$$

Now: want $Q \succeq 0$ for some setting of α

In this case $\alpha = 3$, $Q \succeq 0$. (has eigenvalues 7, .5, 0)

BTW: univariate poly $p(x)$, positive $\Rightarrow$ ~~roots~~ roots are either 0 or complex conjugate

$$\Rightarrow p(x) = x^c \cdot \prod_{a \in S} (x-a)(x-\bar{a}) = x^c \prod_{a \in S} (x^2 + a^2)$$

c is even
else ~~$p(\infty) \to -\infty$~~
$p(-\infty) \to -\infty$

multiply it out: sum of squares.

Grothendieck's Inequality:

Given numbers $a_{ij} \in \mathbb{R}$ (both positive or negative) want to figure out how.

$$OPT := \max_{\substack{x_i \in \{\pm 1\} \\ y_j \in \{\pm 1\}}} \sum a_{ij} x_i y_j$$

relates to

$$SDP := \max_{\substack{\|u_i\|_2 = 1 \\ \|v_j\|_2 = 1}} \sum a_{ij} \langle u_i, v_j \rangle$$ ← the SDP relaxation.

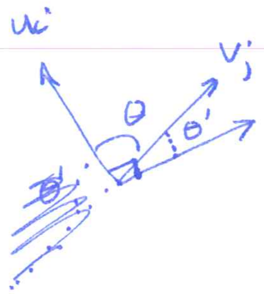
. **Fact**: $SDP \geq OPT$ (it is a relaxation)

Thm! $OPT \geq 0.56 \cdot SDP$ [Krivine, can do better...]

Pf: Suppose we try the simplest idea. These vectors u_i, v_j are on unit ball.
So take the random hyperplane, and set $\hat{x}_i = 1$ if $\langle \bar{g} \cdot \bar{u}_i \rangle \geq 0$
$= -1$ if $\langle \bar{g} \cdot \bar{u}_i \rangle < 0$ etc

$$\Rightarrow \hat{x}_i = \text{sign}\langle \bar{g} \cdot \bar{u}_i \rangle, \quad \hat{y}_j = \text{sign}\langle \bar{g} \cdot \bar{v}_j \rangle.$$

then $E[\hat{x}_i \hat{y}_j] = -1 \cdot \Pr(ij \text{ separated}) + 1 \cdot \Pr(ij \text{ not separated})$
$= -1 \cdot \theta/\pi + 1 \cdot (1 - \theta/\pi) = 1 - \frac{2\theta}{\pi} = \frac{2}{\pi}(\pi/2 - \theta)$
$= 2/\pi \cdot \theta'$



$$SDP_{ij} = \langle u_i, v_j \rangle = \cos\theta = \sin\theta'$$

$\cancel{\Rightarrow E[Alg_{GW}] = E[\sum a_{ij} \hat{x}_i \hat{y}_j] = SDP}$

Here's the problem: for Max Cut we were happy to say that $E[Alg_{ij}]$ was bigger than SDP_{ij}. But now we have weights that are $\boxed{\text{negative}}$, so cannot do this!

Krivine's Trick:

Lemma: Given $\underbrace{u_1 \ldots u_n, v_1 \ldots v_m}_{\text{unit}}$, can get $\hat{u}_i, \hat{v}_j$ (unit vectors) such that

$$\frac{\pi}{2} E\Big[\underbrace{\text{sign}(\langle \hat{u}_i, g\rangle)}_{\hat{x}_i} \cdot \underbrace{\text{sign}(\langle \hat{v}_j, g\rangle)}_{\hat{y}_j}\Big] = c \cdot \langle u_i, v_j\rangle$$

equality!!

N.b. LHS $= \frac{\pi}{2} \cdot \text{Alg}_{ij} \Rightarrow E[\text{Alg}] = E\Big[\sum_{ij} a_{ij} \hat{x}_i \hat{y}_j\Big] = \sum_{ij} a_{ij} \cdot \frac{c}{(\pi/2)} \langle u_i, v_j\rangle = \frac{c}{(\pi/2)} \cdot \text{SDP}.$

Also: LHS, by calculation above $= \sin^{-1}(\langle \hat{u}_i, \hat{v}_j\rangle)$.

So we want to prove that $\langle \hat{u}_i, \hat{v}_j\rangle = \sin(c\langle u_i, v_j\rangle)$.

Pf: existence

$$\sin(c \cdot \langle u_i, v_j\rangle) = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{c^{2k+1}}{(2k+1)!} (u_i \cdot v_j)^{2k+1}$$

$$= \sum_{k=0}^{\infty} (-1)^k \cdot \frac{c^{2k+1}}{(2k+1)!} \left(u_i^{\otimes 2k+1} \cdot v_j^{\otimes 2k+1}\right)$$

$(u \cdot v)^2 = u^{\otimes 2} \cdot v^{\otimes 2}$ etc.

So define $\hat{u}_i$ to be a vector whose k^{th} set of coordinates are $(-1)^k \cdot \sqrt{\frac{c^{2k+1}}{(2k+1)!}} \cdot u_i^{\otimes 2k+1}$

$\hat{v}_j$ has $\sqrt{\frac{c^{2k+1}}{(2k+1)!}}\, v_j^{\otimes 2k+1}$

then by def^n $\langle \hat{u}_i, \hat{v}_j\rangle = \sin(c\langle u_i, v_j\rangle)$.

What are lengths of $\hat{u}_i$? $\|\hat{u}_i\|^2 = \sum_{k=0}^{\infty} \frac{c^{2k+1}}{(2k+1)!} \cdot \|u_i\|^{(2k+1)2} = \sinh(c\|u_i\|^2)$
$= \sinh(c)$

So choose $c = \sinh^{-1}(1) = \ln(1+\sqrt{2})$
$= 1.$

similarly $\|\hat{v}_j\|^2 = \sinh(c) = 1$.

Finally: this was an infinite construction. But this shows $\exists$ vectors that achieve this.

How do you find those vectors? write an SDP for vectors $\tilde{u}_i, \tilde{v}_j$ s.t.

$$\langle \tilde{u}_i \cdot \tilde{v}_j \rangle = \sin(c \cdot \langle u_i, v_j \rangle) \quad \forall\, i,j$$

inputs!

this SDP gives the solution you want.

Further work: this gave the Grothendieck constant to be ≈ 0.56. ← Krivine's bound.

[Alon Makarychev^2. Naor]. Given a general graph, the gap between OPT and SDP

$$OPT = \max_{x_i \in \{\pm 1\}} \sum_{ij \in E}{}' a_{ij} \langle x_i, x_j \rangle \quad vs \quad SDP = \max_{\|v_i\|=1} \sum_{ij \in E}{}' a_{ij} \langle v_i \cdot v_j \rangle$$

same set of vars

is at most $O(\log \vartheta(\bar{G}))$ ← $K(G)$

the Lovasz theta fn on the complement of G.

For bipartite graphs (which are perfect)

$\vartheta(\bar{G}) = \chi(G)$, get constant factor ← we gave explicit bound.

$\leq$ is true in general

(*) They also show that the Krivine bound is not tight, can beat it by a little bit.

Open Question: What is true bound for K_G? Current bounds:-

$$\Omega(\log \omega(G)) \leq K_G \leq O(\log \vartheta(\bar{G}))$$