Matroids $(U, \mathcal{I})$ s.t. (1) downwards closed

(2) if $|A| < |B|$ and $A, B \in \mathcal{I}$ then $\exists x \in B \setminus A$ st $A + x \in \mathcal{I}$.

- indep sets, bases, circuits (= minimal dependent sets), cut (minimal set that intersects every base).
- rank
- examples: uniform matroid, partition matroids, (laminar matroids) graphic, linear
  - : dual of $M = M^*$ whose bases are complements of M's bases
  - : greedy algo optimizes ~~one~~ for max wt matroid.

rank(S) = size of max wt indep set in S. $\Rightarrow$ (rank(S) = |S| iff $S \in \mathcal{I}$).

rank(S) is a submodular function! Also monotone.

Alternate defn:

$r : 2^U \to \mathbb{Z}^+$ is rank fn iff

1. $r(U) \le |U|$
2. r monotone
3. r submodular

Other examples:

- transversal matroid. Bipgraph, elements are LHS, $A \in \mathcal{I}$ if $\exists$ a matching covering A.

- gammoid

  (V, E) digraph. $S, T \subseteq V$

  $A \subseteq S$ indep if $\exists B \subseteq T$ st $|A| = |B|$ and $\exists$ node disjoint paths from $A \to B$.

  (S may intersect T, so A may intersect B. $\Rightarrow$ 0-length paths are OK.)

- representable matroid: isomorphic to linear matroid over some Field $\mathbb{F}$.
  - binary matroid: if repr over $\mathbb{F}_2$.
  - regular matroid: if over every field $\mathbb{F}$.
  - graphic are regular matroids.

Matroid Polytope:

$$\max \sum w_e x_e$$

$$\text{s.t. } x(S) \le r(S) \quad \forall \text{ sets } S$$

$$x \ge 0.$$

Claim: the matroid polytope is integral, solvable by greedy algorithm

Big breakthrough: Matroid Intersection.

Same groundset U, two matroids $M_1 = (U, \mathcal{I}_1)$, $M_2 = (U_2, \mathcal{I}_2)$.

a common indset $S \in \mathcal{I}_1 \cap \mathcal{I}_2$. want max cardinality i.s. (or max wt i.s.).

Thm: $\exists$ a polytime algo to find a common i.s. of max cardinality (weight).

Thm: $$\max_{S \in \mathcal{I}_1 \cap \mathcal{I}_2} |S| = \min_{X \subseteq U} \left(r_1(X) + r_2(U \setminus X) \right)$$

Pf: ($\le$) for any $S \in \mathcal{I}_1 \cap \mathcal{I}_2$ $$|S| = |S \cap X| + |S \setminus X| \le r_1(X) + r_2(U \setminus X).$$

($\ge$) we prove algorithmically soon.

Thm: Max cardinality common indset in 3 matroids is NP hard.

Examples:
- bipartite matching
- min cost arborescence.
- colorful spanning tree (one with every edge having a different color).
- two disjoint spanning trees (intersection of graphic M and co-graphic M^*).
- k disjoint spanning trees?

To prove this Matroid Intersection theorem, recall two facts you proved:-

- given $X \in \mathcal{I}$ define graph

$H_M(X) =$ 

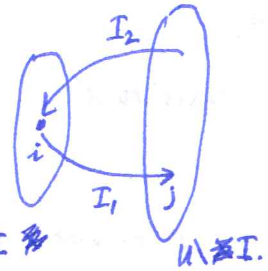
with edge i,j if $X - i + j \in \mathcal{I}$

then ① if $|Y| = |X|$ and $Y \in \mathcal{I}$ then $\exists$ a perfect matching on $X \Delta Y$.

② if $\exists$ Y st unique perfect matching on $X \Delta Y$ then $Y \in \mathcal{I}$.

Now the Algorithm:

Given M_1, M_2 and $I \in \mathcal{I}_1 \cap \mathcal{I}_2$, define <u>directed</u> graph $H_{M_1,M_2}(I)$



(ij) if $I - i + j \in \mathcal{I}_1$

(ji) if $I - i + j \in \mathcal{I}_2$

directed versions of $H_{M_1}(I)$ $\longrightarrow$

$H_{M_2}(I)$ $\longleftarrow$

pasted together.

<u>Algorithm</u>: Let $A_1 = \{ a \in U \setminus I \mid I + a \in \mathcal{I}_1 \}$

$A_2 = \{ a \in U \setminus I \mid I + a \in \mathcal{I}_2 \}$.

<u>Note</u>: if $\exists\, a \in A_1 \cap A_2$ then $I + a$ is a larger ind set in $\mathcal{I}_1 \cap \mathcal{I}_2$.

so can assume that $A_1 \cap A_2 = \emptyset$.

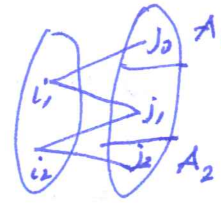
<u>Lemma 1</u>: if $\exists$ no $A_1 \to A_2$ path in $H_{M_1,M_2}(I)$ then I is a max card indset in $\mathcal{I}_1 \cap \mathcal{I}_2$

<u>Lemma 2</u>: if P is a <u>shortest</u> $A_1 \to A_2$ path then $I \Delta \underset{\text{vertices}}{P} \in \mathcal{I}_1 \cap \mathcal{I}_2$.

So algorithm is clear: keep finding $A_1 \to A_2$ paths, (rebuilding $H_{M_1,M_2}(I)$ after each step) until no more. that is max card indset in $\mathcal{I}_1 \cap \mathcal{I}_2$.

<u>Proofs</u>: (Lemma 2)

Sps $j_0, j_1 \ldots j_k$ added $i_1, i_2 \ldots i_k$ removed. show this in $\mathcal{I}_1$ (Similarly for $\mathcal{I}_2$)



define $J = I - (i_1, \ldots i_k) + (j_1, \ldots j_k)$.

(1) $|J| = |I|$. and $\exists$ unique p.m in $J \Delta I$, so $J \in \mathcal{I}_1$.

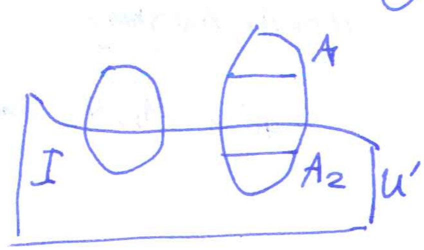
(2) each $j_t \notin A_1 \Rightarrow j_t \in \text{span}(I) \Rightarrow \text{rank}_1(I \cup J) = |I| = |J|$

(3) $j_0 \in A_1 \Rightarrow$ $\text{rank}_1(I \cup j_0) \geq |I| + 1$.

$\text{rank}_1(I \cup J \cup j_0) \geq |I| + 1$ so j_0 must increase the rank, it is not spanned by J.

$\Rightarrow J \cup j_0 \in \mathcal{I}_1$

Similarly $I \Delta V(P) \in \mathcal{I}_2$ too and so we've augmented the indset! ☺



Lemma 1: So now $\nexists$ $A_1 \to A_2$ path in H

- $A_1, A_2 \neq \emptyset$ else it is trivial.
- Let U' be set of vertices not reachable from A_1

$A_2 \subseteq U'$. No edge from $U \setminus U'$ to U'

Claim: $|I| \geq r_{M_1}(U') + r_{M_2}(U \setminus U')$. so max-min formula is tight.

(1) $r_1(U') \leq |I \cap U'|$

Pf: sps $r_1(U') > |I \cap U'|$

then $\exists j \in U' \setminus I$ st $(I \cap U') + j \in \mathcal{I}_1$ but I also in $\mathcal{I}_1$ so by matroid axiom,

$\exists Z \in \mathcal{I}_1$ st $(I \cap U') + j \subseteq Z \subseteq I + j$

and $|Z| \geq |I|$

so either $Z = I + j$ (but $j \notin A_1$ contradiction)

or $Z = I - i + j$ (but then arc from top to bottom).
$i \in I \setminus U'$

so contradiction, and hence $r_1(U') \leq |I \cap U'|$.

(2) $r_2(U \setminus U')$ similarly $\leq |I \setminus U'|$.

So $|I| \geq r_1(U' \cap I) + r_2(U \setminus U')$

$\Rightarrow$ equality and hence it is a max cardinality set in $\mathcal{I}_1 \cap \mathcal{I}_2$

For bip matchings, each arc is an potential edge flip, so each aug path in H is really like an aug path in the original graph.

Max wt: ~~pick~~ set vertex costs $= +w_e$ for $e \in I$
$-w_e$ for $e \notin I$

find min cost path (and among them, path of min # hops) from $A_1 \to A_2$.