Max Flows using Electrical Flows

Basics: We're given an electrical network modeled as an undirected graph $G=(V,E)$.

Each edge has a resistance. R_e

- If we set voltages of 1 at s and 0 at t, and let the voltages at other points "float" this will result in electrical flow from s to t.
- **Note:** if voltages are ϕ_v then current on $\vec{e}(u,v)$ is $\left(\frac{\phi_u - \phi_v}{R_e}\right) = f_e$ Ohm's Law
- The flow will follow Kirchoff's Laws and will minimize the energy

$$\mathcal{E}(\text{edge}) = f_e^2 R_e = \frac{(\phi_u - \phi_v)^2}{R_e} \qquad \text{total energy} = \sum_e \mathcal{E}(f_e)$$

$$\mathcal{E}(f) = \sum_{u \sim v} \frac{(\phi_u - \phi_v)^2}{R_e}.$$

Algo: • Can find this by solving a linear system:-

Define the Laplacian of edge $= \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = L_e$ ($n \times n$) and $L_G = \sum_{e \in G} L_e$

$= \text{diag}(\text{deg vector}) - A.$

For a weighted (by resistances) graph set $L_G = \sum_e \frac{1}{R_e} L_e.$

Note: $Lx = \left(\sum_{v \sim u} \frac{x_u - x_v}{R_e}\right)_{u \in U}$

If we set $Lx = \text{excess vector} \leftarrow b$ how much you want flow out of each vertex

and solve this for x, get the corresponding voltages (potentials)

And hence the flow.

So want to solve $Lx = F(\chi_s - \chi_t).$

Thm: [Spielman Teng, Koutis Miller Peng] ....

∃ an algorithm to solve linear systems of Laplacians, upto $(1+\epsilon)$ accuracy in time $\tilde{O}(m)$.

[For today, we ignore the precision issue, imagine we can solve the linear system exactly in $\tilde{O}(m)$ time.]

- flow better than the shortest path?

it does not overload edges as much; spreads it out.

Formally: remember the framework from last time. We want to push F flow from $s \to t$ to achieve two criteria.

(1) if $\exists$ a flow $\tilde{f}$ that satisfies $\sum_i q_e \tilde{f}_e \le 1$

then want to find some f st $\sum_i q_e f_e \le (1+O(\epsilon))$. ~~$\le (1+\epsilon)$~~

approx oracle.

(2) want width $= \max_e f_e$ to be small

Thm: Suppose we set $R_e = q_e + \epsilon/m$ on each edge and get min-energy flow f

then get (1) and (2) with $\rho = \tilde{O}_\epsilon(\sqrt{m})$.

Pf: Since $\exists$ a flow $\tilde{f}$ feasible st $\tilde{f}_e \le 1 \; \forall e \Rightarrow \sum q_e \tilde{f}_e \le 1$.

$\therefore \mathcal{E}(\tilde{f}) \le \sum_i \tilde{f}_e^2 R_e \le \sum_i R_e \le (1+\epsilon)$

$\Rightarrow$ min energy flow has also $\mathcal{E}(f) \le 1+\epsilon$.

$\Rightarrow (\sum_i q_e f_e) \le \sqrt{(\sum q_e f_e^2)(\sum q_e)}$ by C-S

$\le \sqrt{1+\epsilon} \le 1+\epsilon$. (1)

and $\frac{f_e^2 \cdot \epsilon}{m} \le 1+\epsilon = \mathcal{E}(f) \Rightarrow f_e^2 \le \sqrt{\frac{(1+\epsilon)m}{\epsilon}} \le \sqrt{2m/\epsilon}$.

$\Rightarrow$ Now MW ~~gives~~ uses $O\left(\frac{\rho \log m}{\epsilon^2}\right)$ iterations of this subroutine

$\Rightarrow \tilde{O}(\sqrt{m}) \times \tilde{O}(m)$ per oracle call $\Rightarrow \tilde{O}(m^{3/2})$.

Pretty good! But actually Even-Tarjan [1975] does $\tilde{O}(m^{3/2})$ too.

So how to do better.

Let us try to do better now. But: have to open black box, and maintain weights explicitly.

<u>Idea</u>: if e has too much flow, kill the edge for all future times.

Like setting the weight to ∞.

Important quantity: the effective resistance R_{eff}^{st} (shorthand R_{eff}) between s & t. R_{eff} = Energy when we send unit flow ~~from~~ from s to t.

= 1/(Energy when we maintain a potential diff of 1 b/w s & t).

<u>Claim</u>: Suppose we change the edge resistances from R_e to R_e'.

(a) if $R_e' \geq R_e \Rightarrow R_{eff}' \geq R_{eff}$.

(b) if for some $s \to t$ flow f if e is such that $f_e^2 R_e \geq \beta \cdot \mathcal{E}(f)$ then deleting e means $R_{eff}' \geq R_{eff}/(1-\beta)$.

Pf: ~~later~~ (a) for each flow f, $\mathcal{E}(f) = \sum f_e^2 R_e \leq \sum f_e^2 R_e' = \mathcal{E}'(f)$

$\Rightarrow \min_f \mathcal{E}(f) \leq \min_f \mathcal{E}'(f). \Leftrightarrow R_{eff} \leq R_{eff}'$.

(b) $\mathcal{E} = \dfrac{1}{R_{eff}} = \displaystyle\sum_{uv \in E \setminus e} \frac{(\phi_u - \phi_v)^2}{R_{uv}} + \frac{(\phi_u-\phi_v)^2}{R_e} \leftarrow \geq \frac{\beta}{R}$

setting $\phi_s = 1$ $\phi_t = 0$.

take same potentials in $G \setminus e$ then energy $\leq \dfrac{1-\beta}{R_{eff}} \Rightarrow R_{eff}' \geq \dfrac{R_{eff}}{1-\beta}$.

∎

<u>Good</u>: everytime we remove an edge with a lot of energy, the R_{eff} jumps!

$$\boxed{\rho = \frac{m^{1/3} \ln m}{\epsilon}}$$

<u>Also</u>: start with $w_e = 1$.

- Each time set $R_e^t = w_e^t + \dfrac{\epsilon W^t}{m}$. find flow f^t.

  if $\exists$ edge with $f_e^t > \rho$ then delete edge and repeat.

  else use the flow, update $w_e^{t+1} \leftarrow w_e^t \left(1 + \frac{\epsilon}{\rho} \cdot f_e^t\right)$.

<u>Important</u>: assume that $F \geq \Theta\left(\underbrace{\frac{m^{1/3} \log m}{\epsilon^2}}_{=\rho}\right)$ else can just use FF to solve the problem.

Claim: the algorithm deletes at most $D \le \tilde{O}(m^{1/3})$ edges
and hence removes at most $\le \epsilon F$ flow from graph.

$$\text{the runtime} = \tilde{O}\left(Dm + \frac{\rho \log m}{\epsilon^2}\right) = \tilde{O}_\epsilon(m^{4/3}).$$

Pf: We track the Reff, and also W^t over the algorithm.

Reff: starts off with a graph where maxflow = OPT
$\Rightarrow$ mincut = OPT $\Rightarrow$ $\exists$ edge in this cut that carries $\geq \frac{1}{OPT}$ flow in any unit s-t flow

$$\Rightarrow \text{Reff} = \text{energy of this flow} \geq \frac{w_e + \epsilon W/m}{OPT^2} \geq \frac{1}{OPT^2}.$$

$W^0 = m.$

~~to each~~ As long as we remove at most ϵF ~~flow~~ edges, the graph can still support F flow with congestion $\le \frac{1}{1-\epsilon}$ and hence has total energy

$$\leq \sum_e r_e f_e^2 \leq \frac{1}{(1-\epsilon)^2} \sum_e \left(w_e^t + \frac{\epsilon W^t}{m}\right) \leq \frac{1+\epsilon}{(1-\epsilon)^2} W^t \leq 2W^t.$$

$\Rightarrow$ min energy flow $\le 2W^t$ (energy)

In each iteration (successful),

$$W^{t+1} = \sum_e w_e^t\left(1 + \frac{\epsilon}{\rho} f_e^t\right) \leq W^t + \frac{\epsilon}{\rho} \cdot \overbrace{\sum_e w_e^t f_e^t}^{\text{C-S as before}} \leq W^t\left(1 + \frac{2\epsilon}{\rho}\right) \text{ (say)}$$

$$\Rightarrow \text{after } T = \frac{\rho \ln m}{\epsilon^2} \text{ rounds } \boxed{W^{T+1} \leq (1+\epsilon) m \cdot e^{+2\ln m/\epsilon}}$$

- ~~After~~ Any edge removed has energy $\geq \rho^2 \cdot \epsilon W^t/m$
  total energy at that time $\le 2W^t$ $\Rightarrow$ Reff goes up by $\left(1 - \frac{\epsilon \rho^2}{2m}\right)^{-1}$
- And when weights updated, Reff $\uparrow$.

$\Rightarrow$ after D deletions, and any ~~#~~ successful iters

$$R_{eff}^{(D)} \geq \frac{R_{eff}^{(0)}}{\left(1 - \frac{\epsilon \rho^2}{2m}\right)^D}.$$

But ~~[illegible]~~ as long as ~~[illegible]~~ D has stayed below $\tilde{O}(m^{1/3}), \leq \epsilon F$, (5)

$\underset{\text{of } F \text{ flow}}{\text{energy}} \leq 2W^t. \Rightarrow \underset{\text{unit flow}}{\text{energy of}} = R_{eff}^D \leq \frac{2W^t}{F^2}.$

$$\Rightarrow \frac{R_{eff}^0}{\left(1 - \frac{\epsilon \rho^2}{2m}\right)^D} \leq R_{eff}^D \leq \frac{(1+\epsilon)m.e^{+\frac{2\ln m}{\epsilon}}}{F^2} \leq 2me^{-2\ln m/\epsilon}$$

Approximating: $\frac{1}{(OPT)^2} e^{+\frac{\epsilon\rho^2.D}{2m}} \leq \cancel{\ } 2m.e^{-2\ln m/\epsilon}$ $OPT \leq m.$

(maybe cheating abit)

$$\Rightarrow \frac{+\epsilon\rho^2 D}{2m} \leq \ln(2m^3) + \frac{2\ln m}{\epsilon}$$

$$\Rightarrow D \leq \frac{2m}{\epsilon\rho^2} \cdot \left[\frac{O(\ln m)(1+\epsilon)}{\epsilon}\right] \leq O(m^{1/3}) \leq \epsilon F.$$

not really

$\frac{1}{1-x} \geq e^{x/2}$ for $x \in [0, 1/4]$ say.

so lose constants here. no big deal