Dynamic Graph Algorithms

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Outline

Dynamic Graph Problems – Quick Intro

Topic 1. (Undirected Graphs)
Dynamic Connectivity & MST

Topic 2. (Undirected/Directed Graphs)
Dynamic Shortest Paths

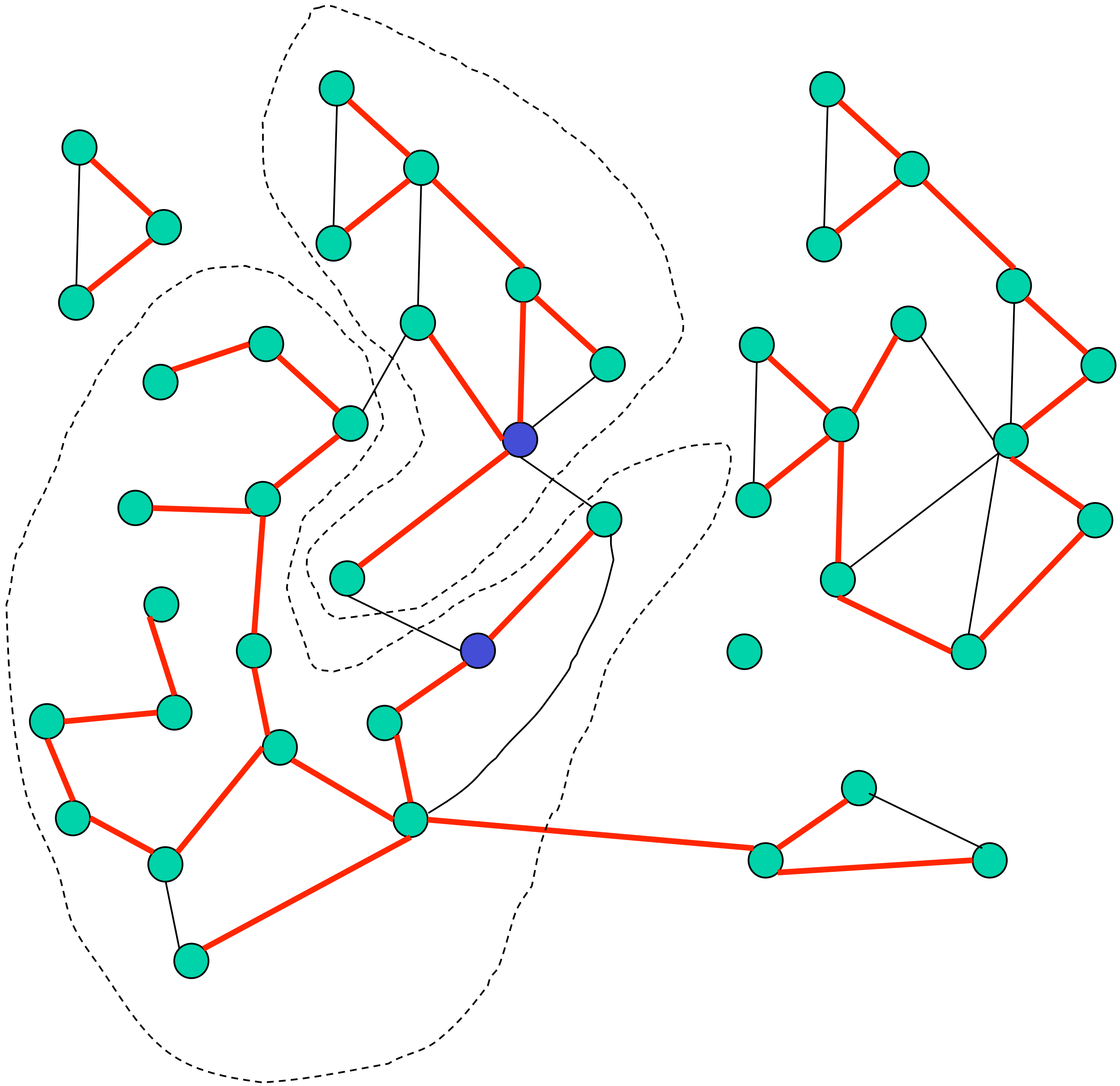
Topic 3. (Non-dynamic?)
2-Connectivity in Directed Graphs

Holm et al.
(Dynamic decomposition)

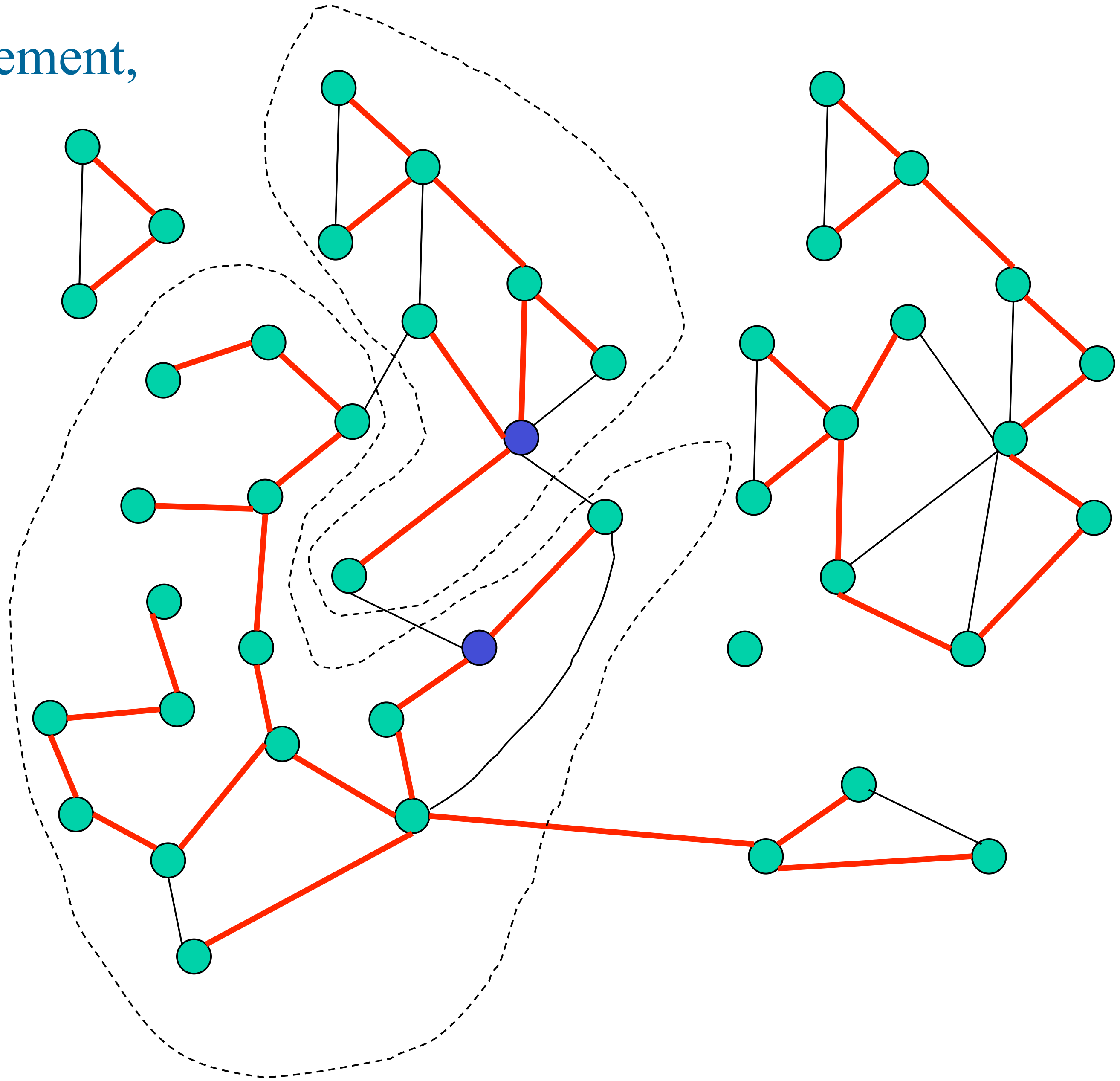
Query	O(log n)
Update	$O(\log^2 n)$

How do we find out whether there is a “replacement” edge for the forest or it really got disconnected ?

For dynamic MSF it is not enough to find a repacement edge, we need to find the best replacement edge



To find a replacement, need to traverse one of the trees, which can be quite expensive.



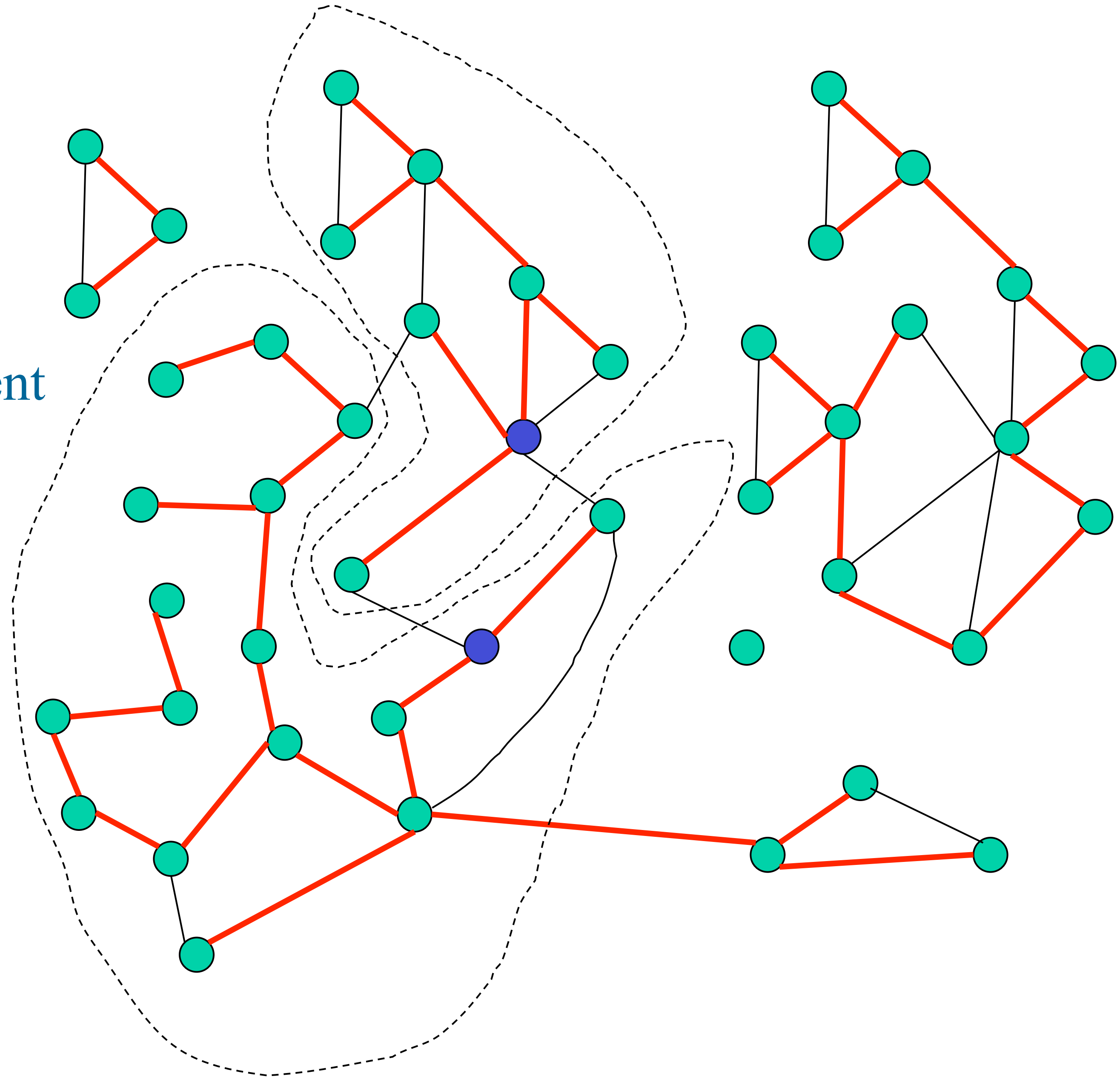
Randomization [Henzinger, King]: sample non-tree edges in smaller tree

If sampling fails, push "sparse cut" to upper level

Can we do this deterministically?

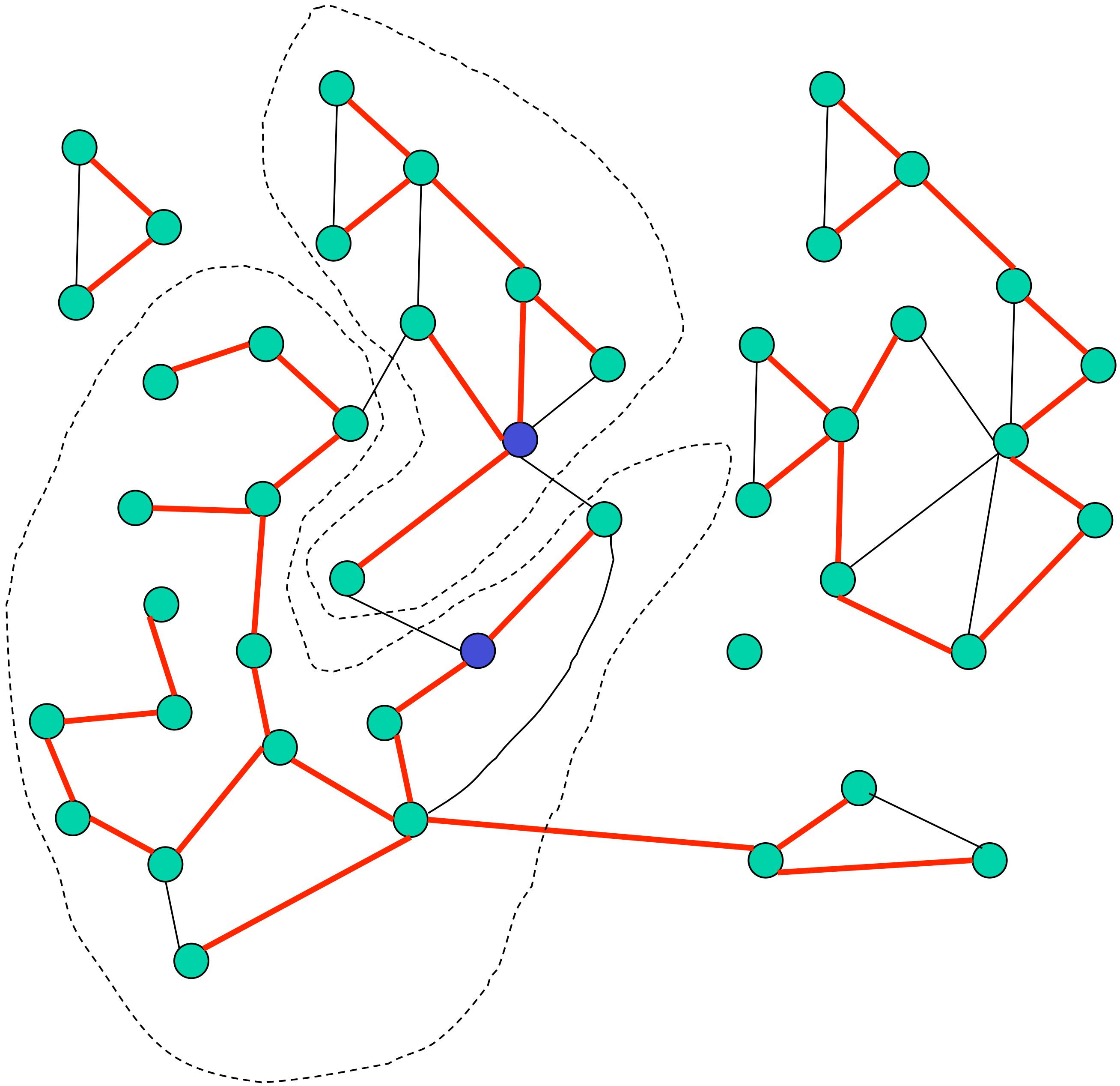
Look in the smaller tree:

- ✗ tree edge
- ✗ no replacement
- ✓ replacement



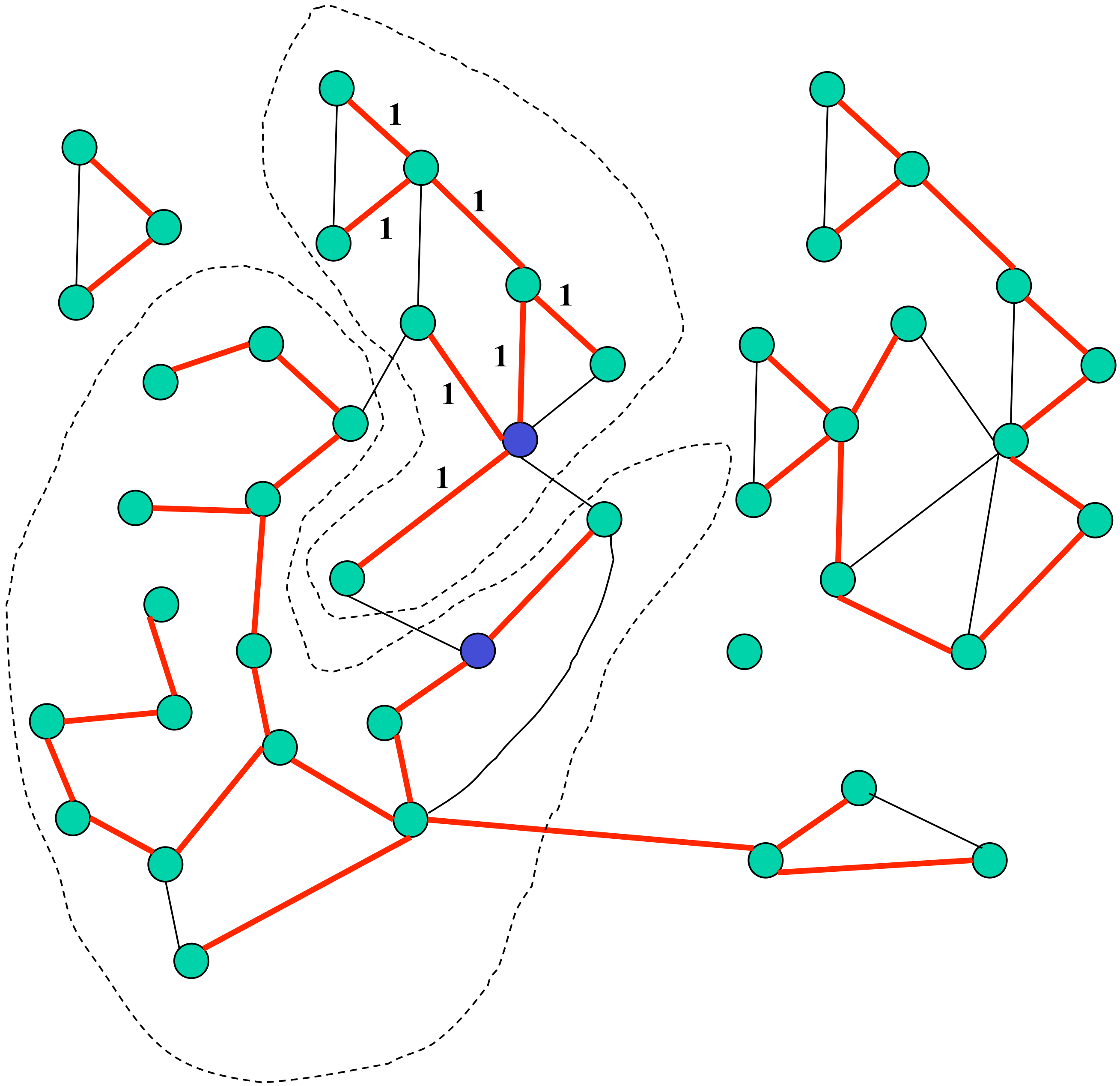
Wish to gain something (in amortized sense) by accumulating information as we do that

Each edge has a level



Each edge has a level

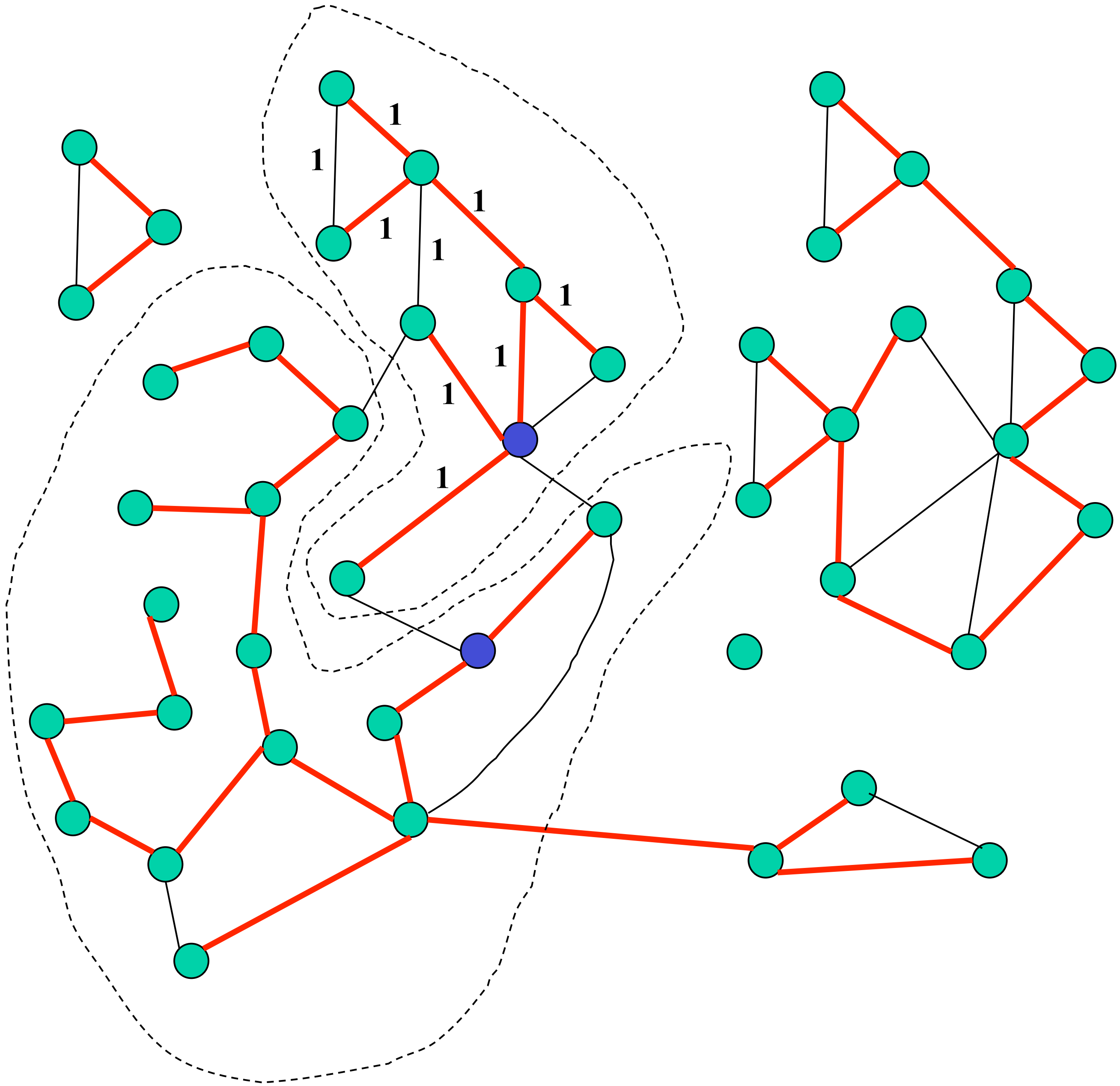
Increase the level of the edges in the smaller tree…



Each edge has a level

Increase the level of the edges in the smaller tree…

… and of any edge discovered not to be a "replacement"

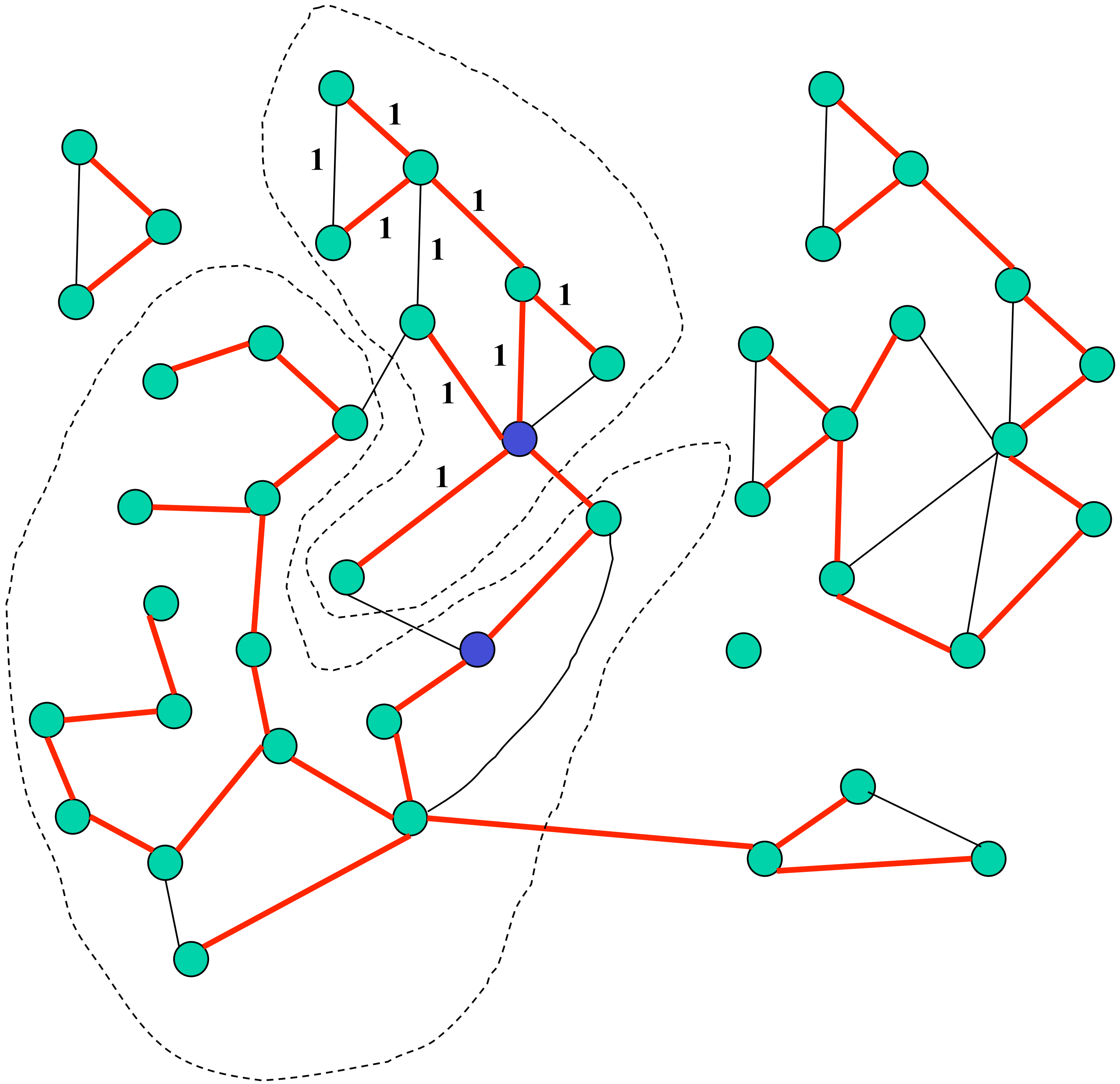


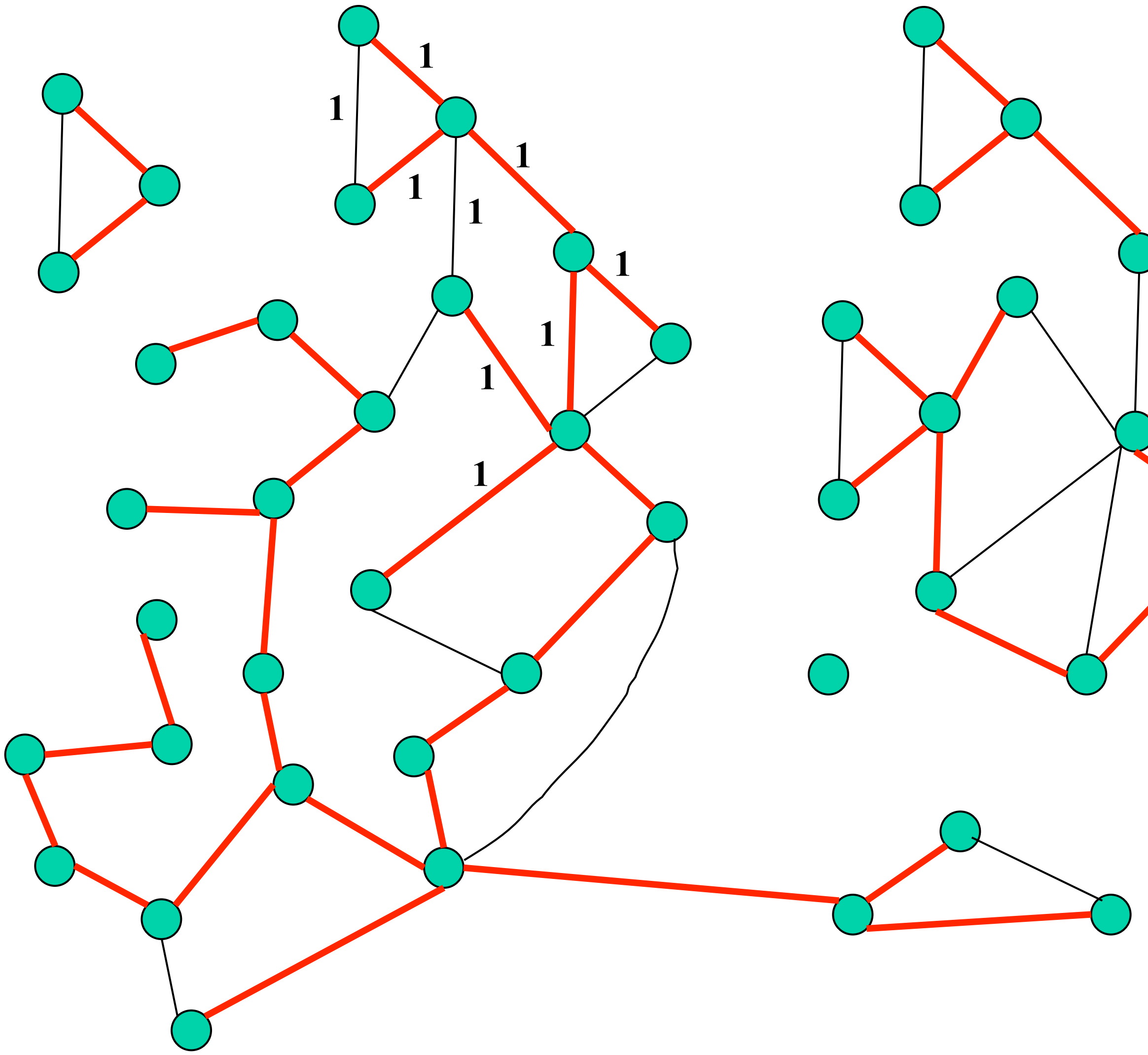
Each edge has a level

Increase the level of the edges in the smaller tree…

… and of any edge discovered not to be a "replacement"

until you find a "replacement"

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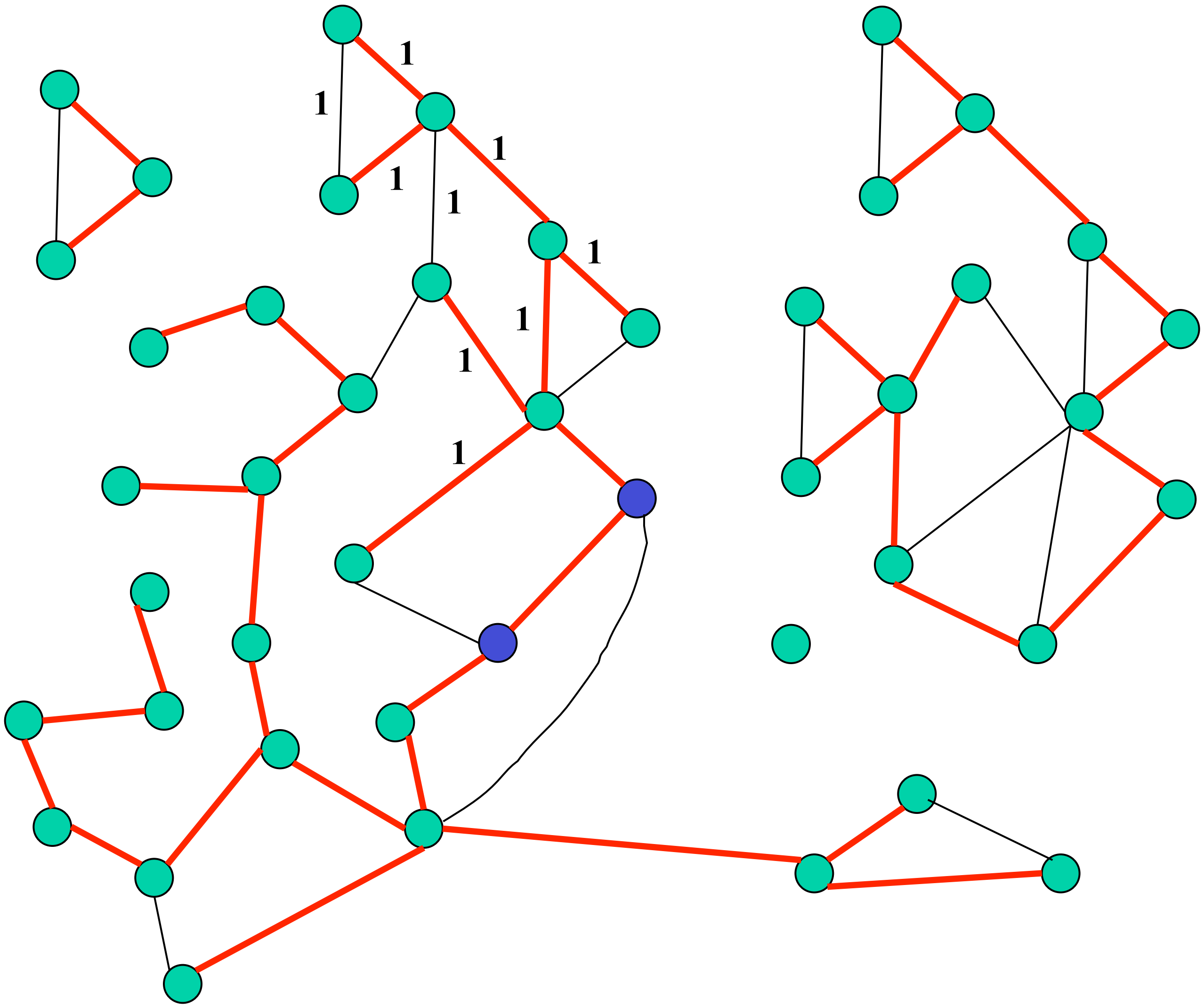
Intuition:
Next time you have to look again for a replacement…



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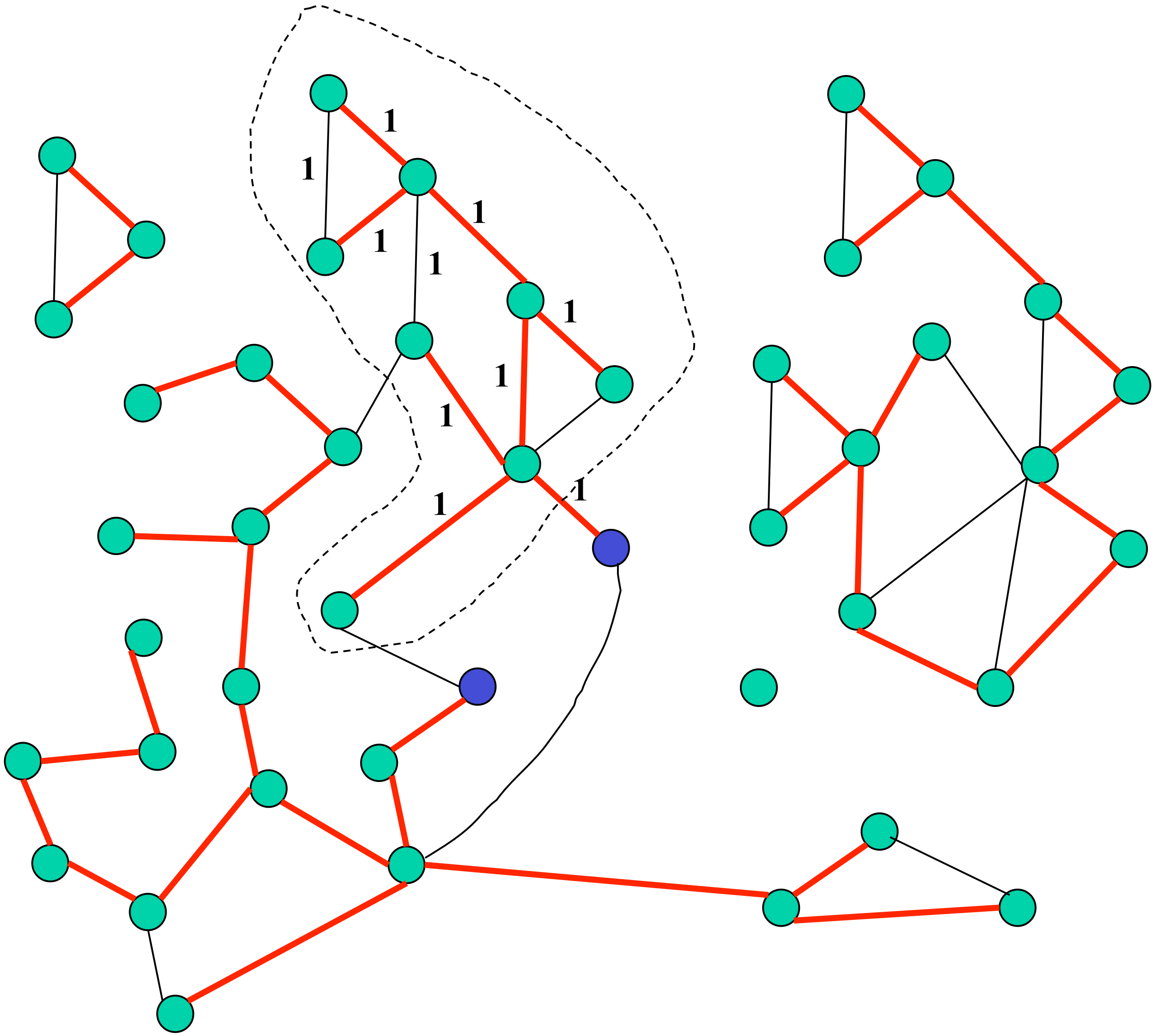

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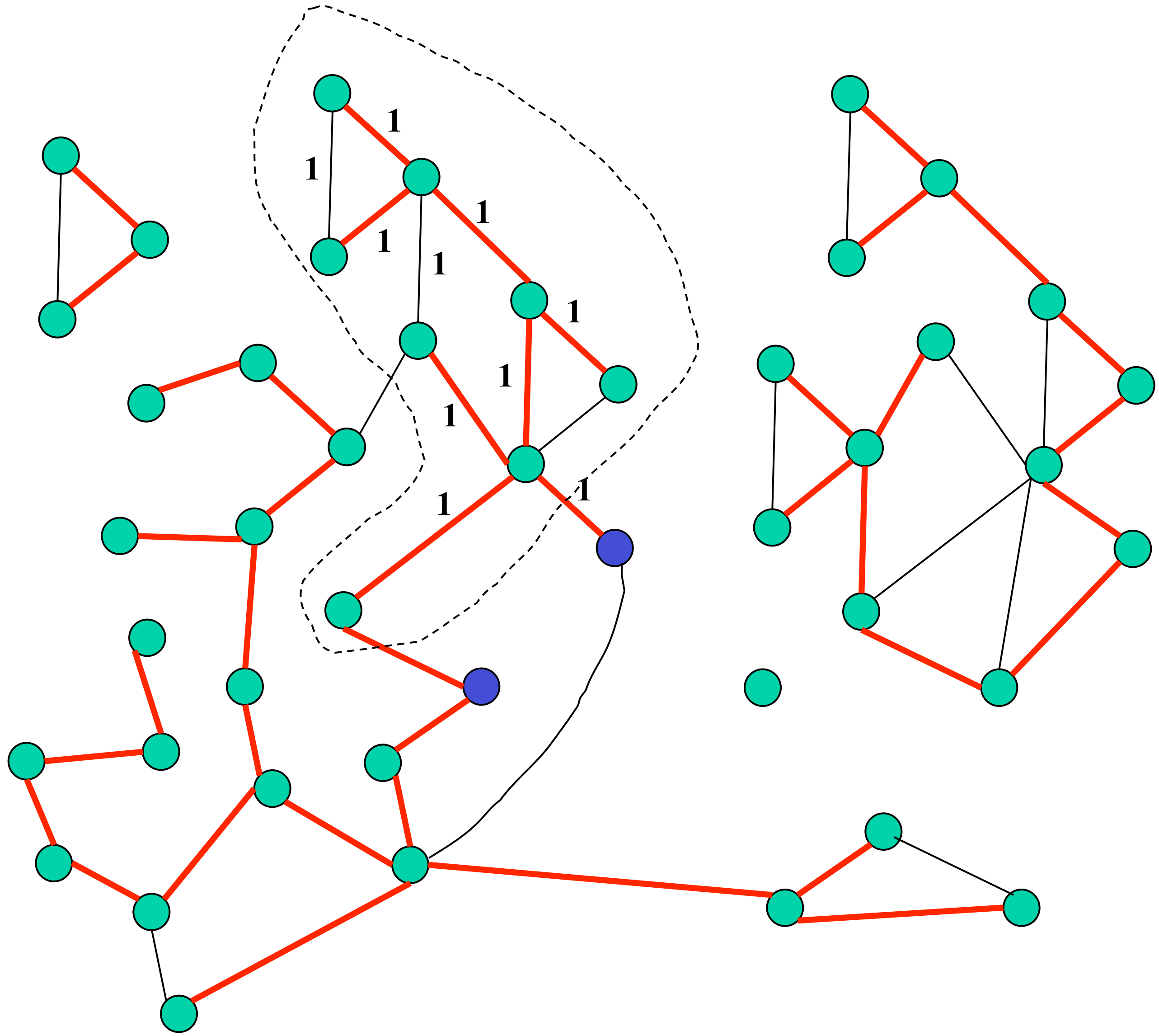

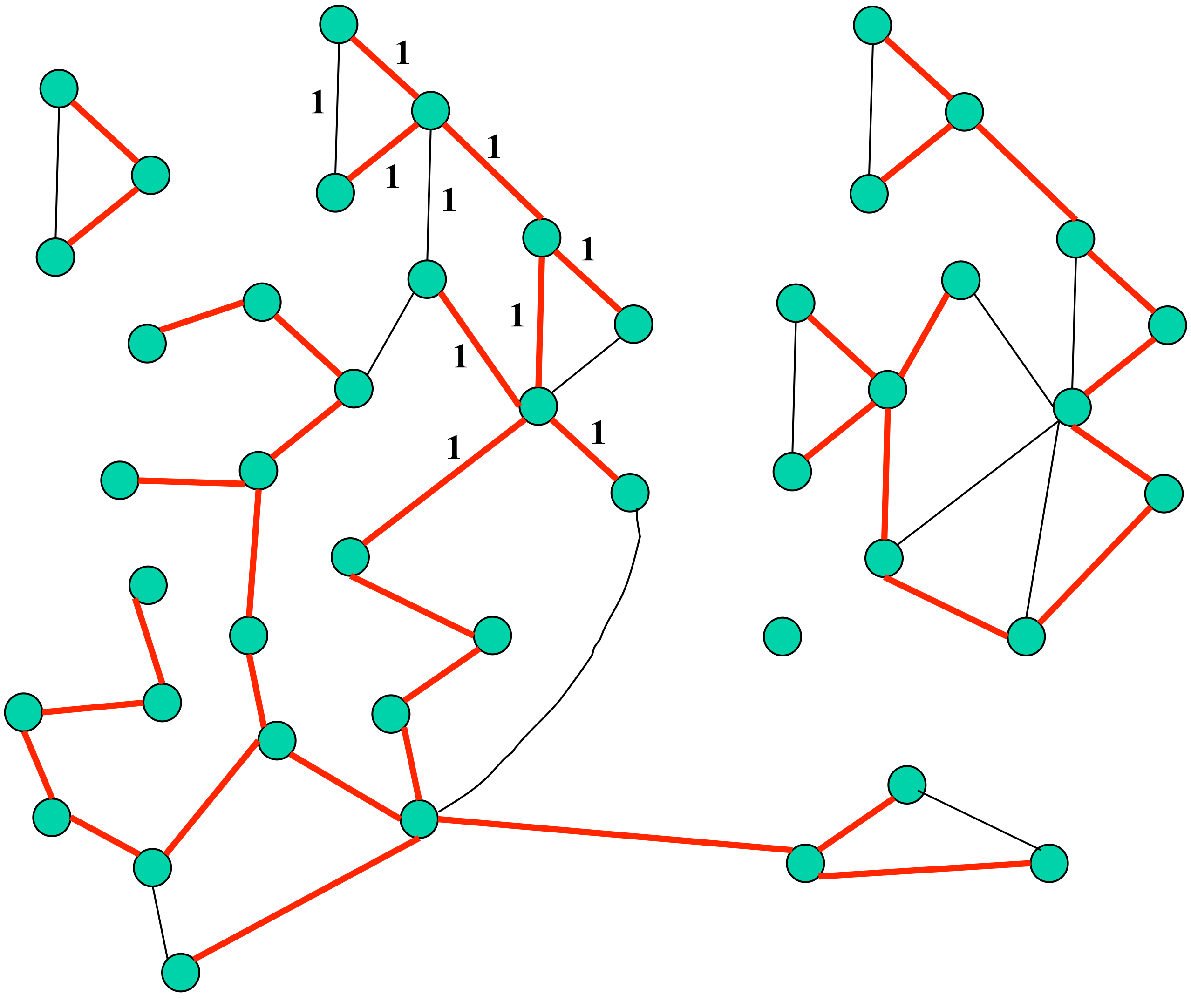
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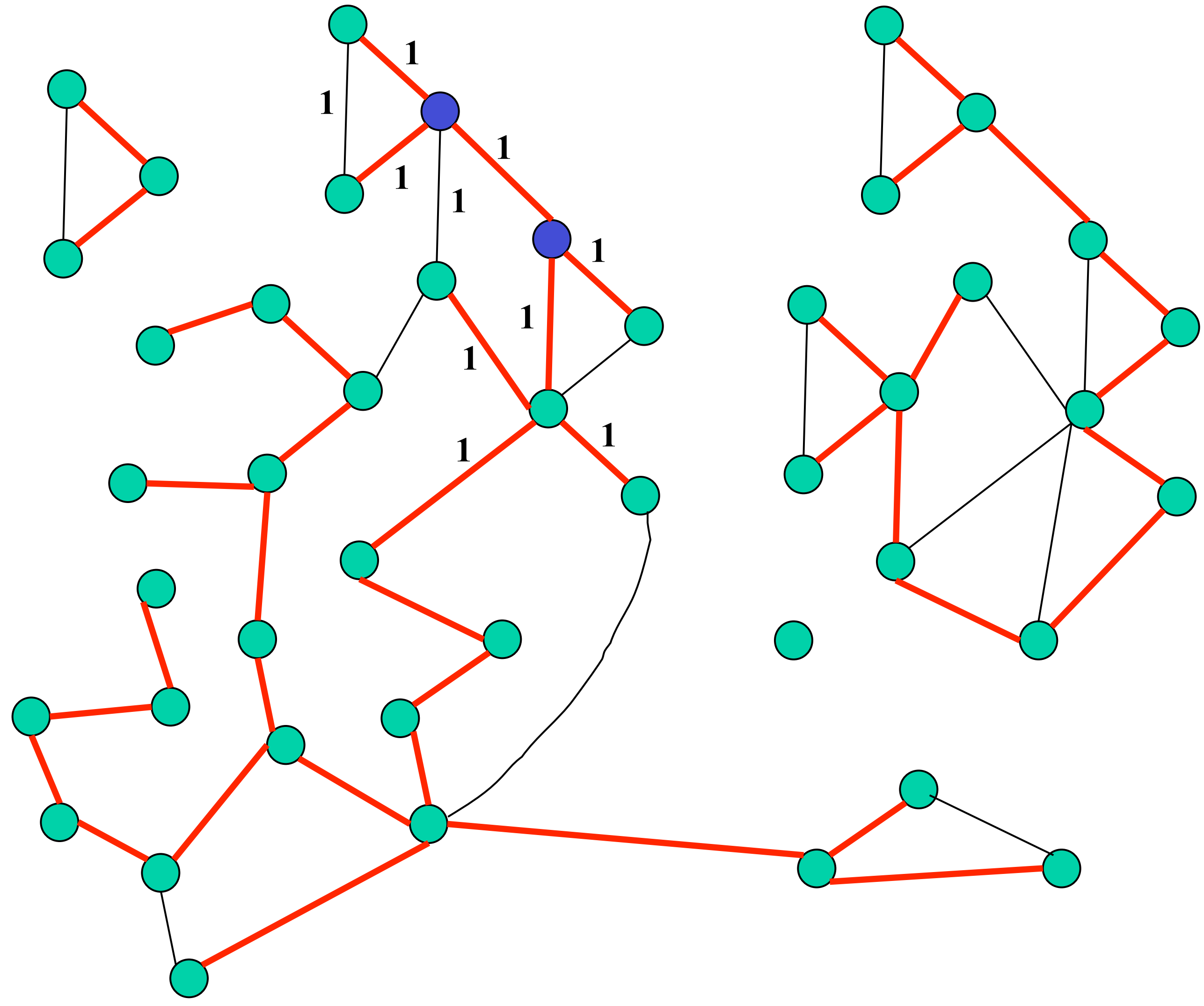
Keep on doing that upon edge deletions



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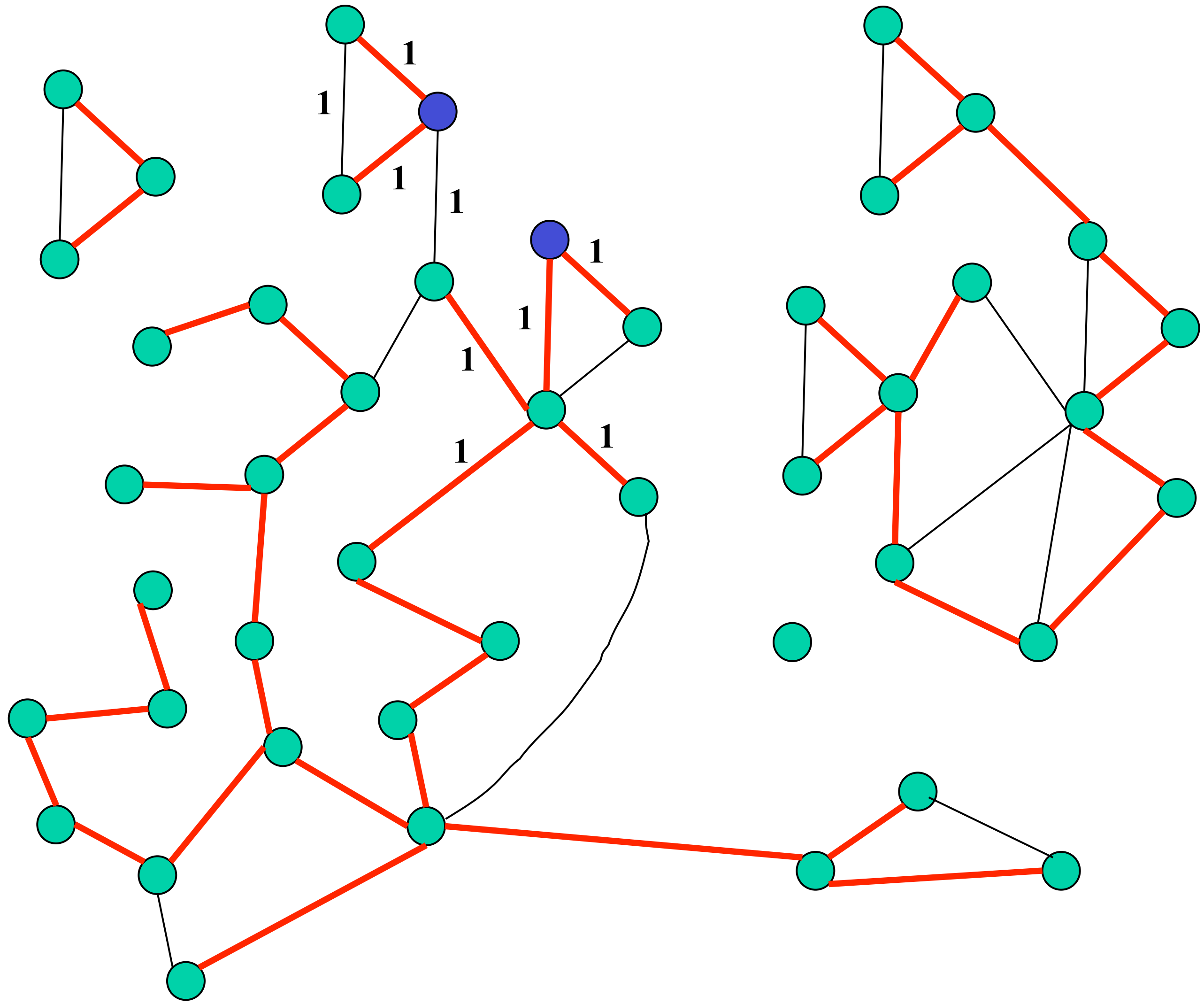

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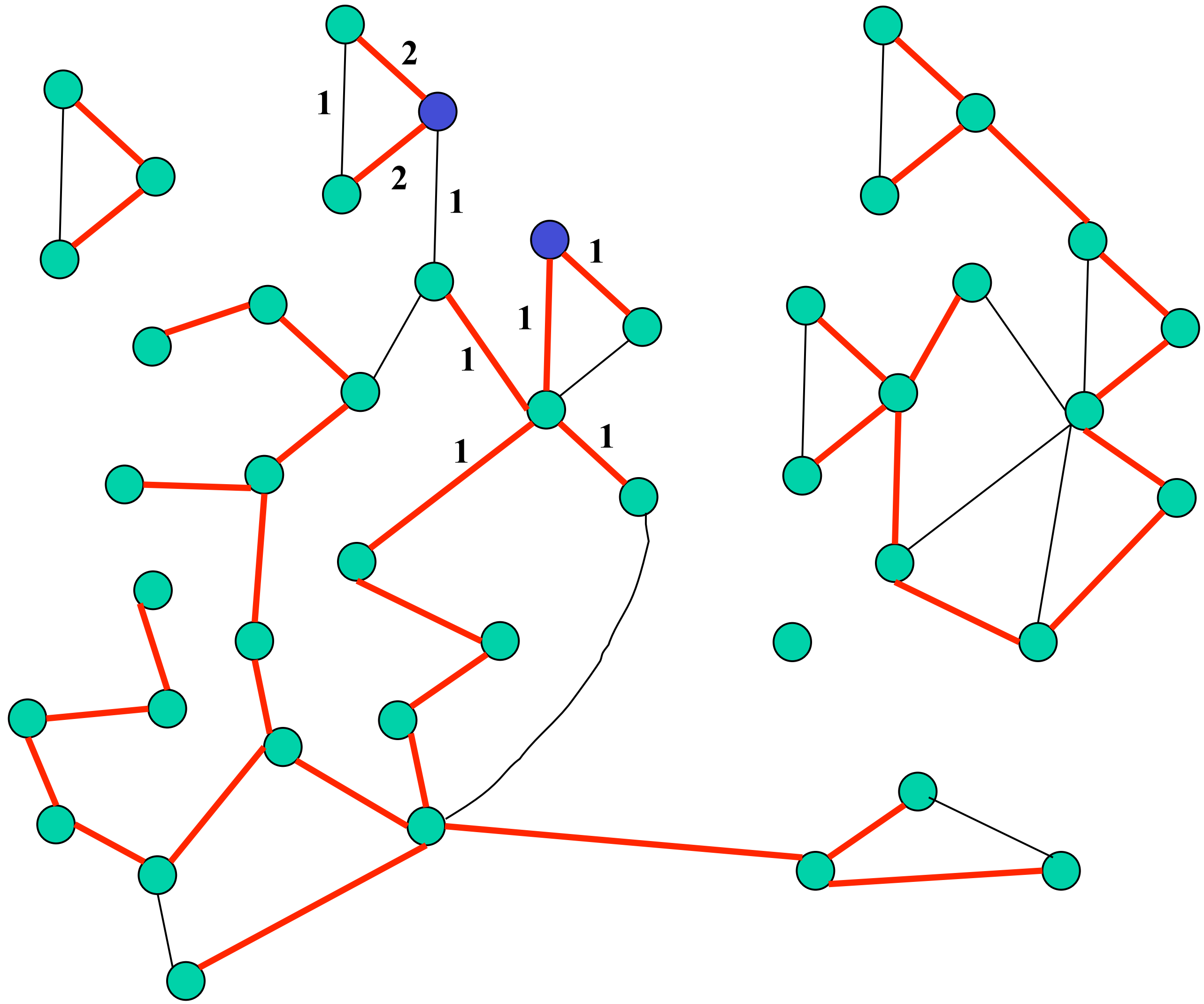
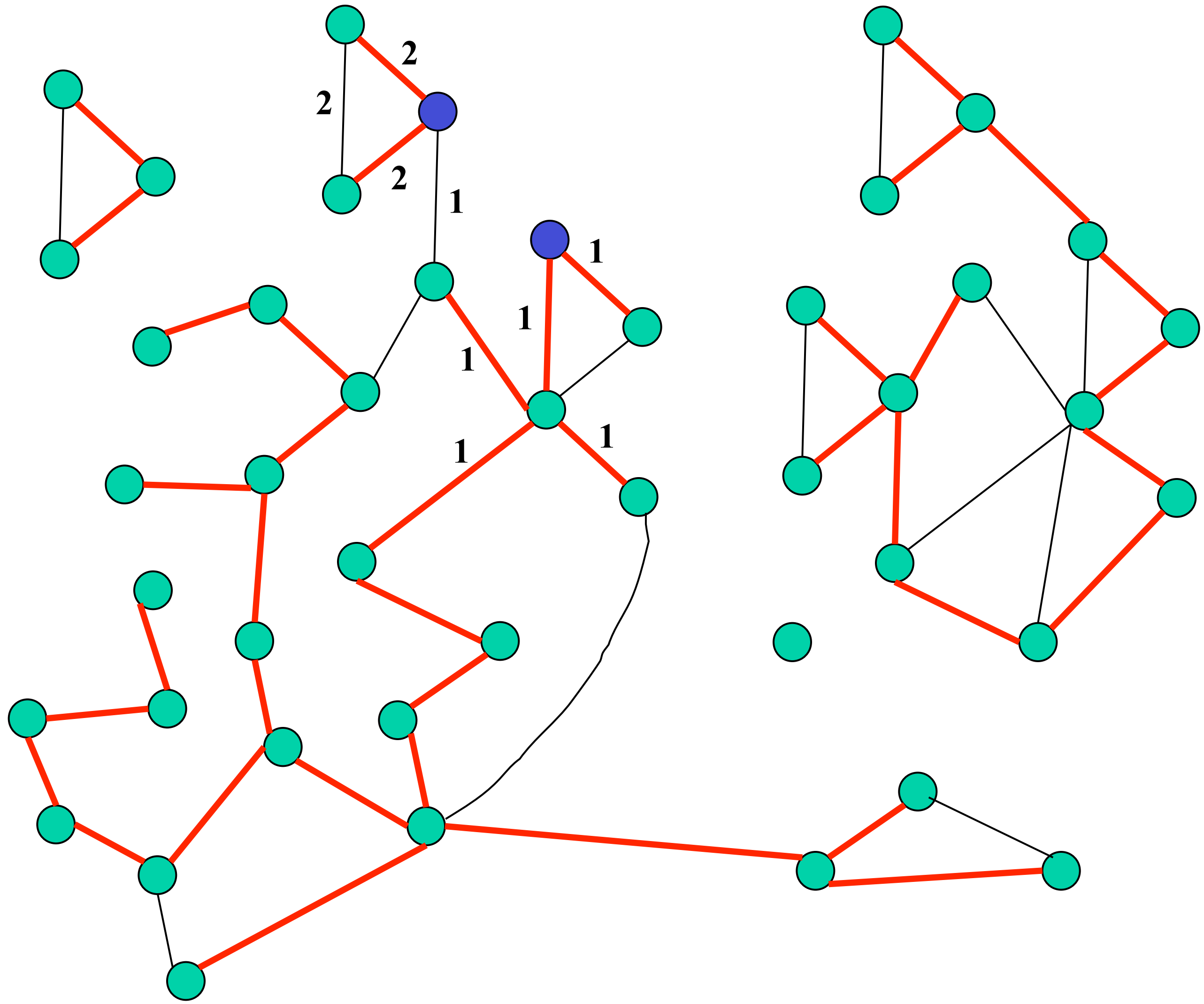


Keep on doing that upon edge deletions

Again, increase the level of the edges in the smaller tree…

… and of any edge discovered not to be a "replacement"



Keep on doing that upon edge deletions

Again, increase the level of the edges in the smaller tree…

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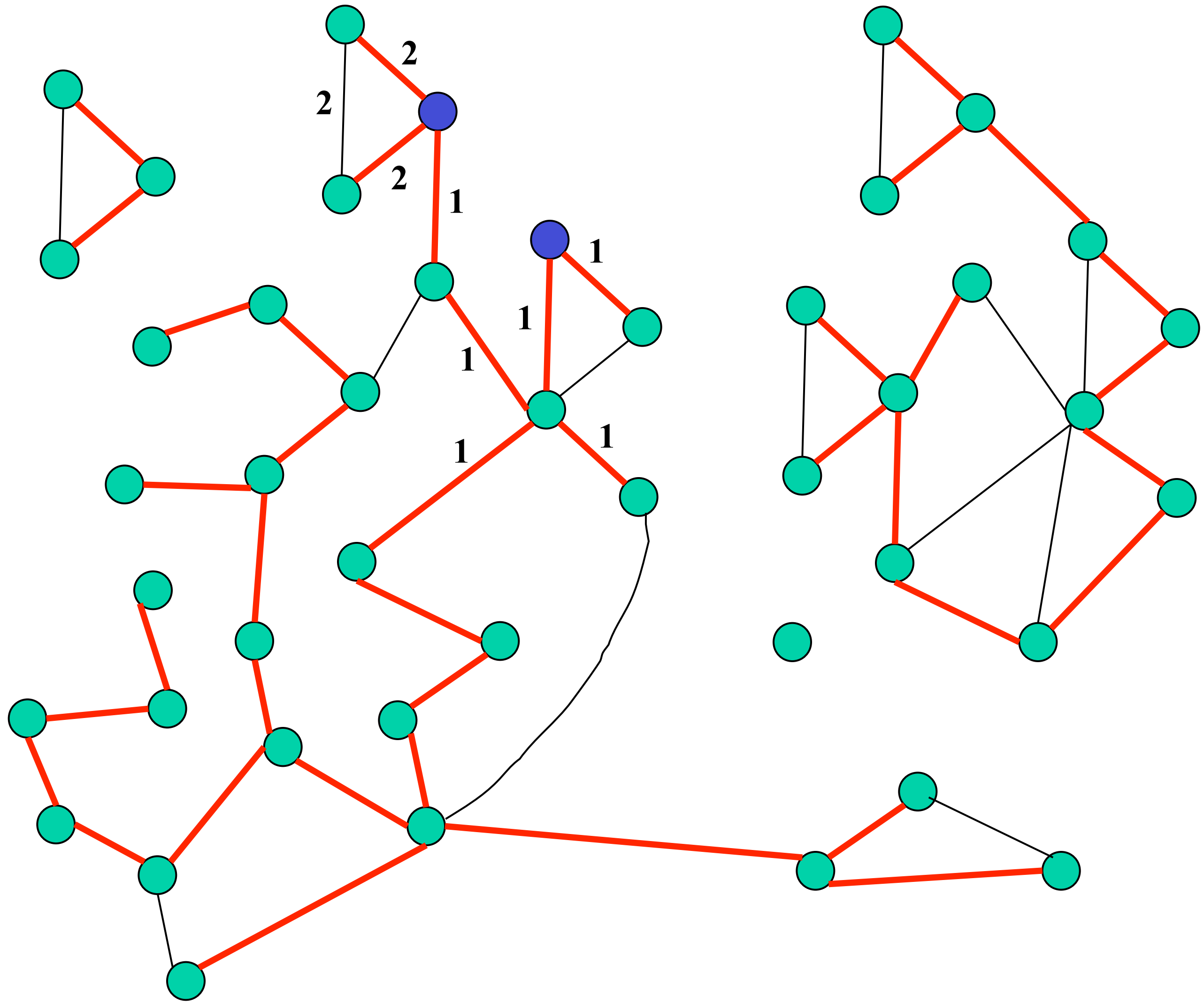
until you find a "replacement"



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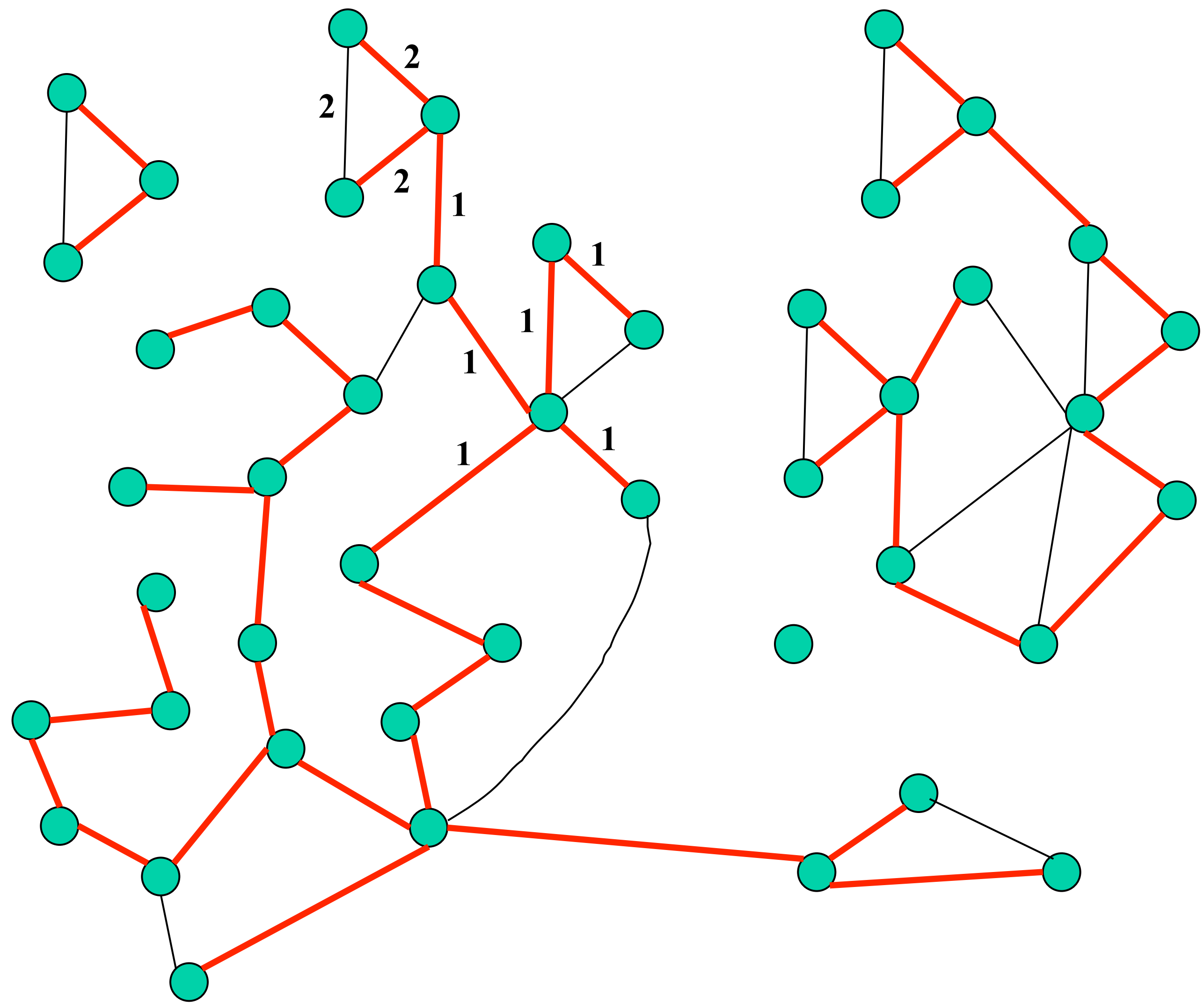


Terminology

G is the dynamic graph. **F** is a spanning forest of G.

An edge is either a tree edge or a non-tree edge.

Each edge has a level ℓ.

$\mathbf{G_\ell}$ is subgraph of G induced by edges of level $\geq \ell$.

$$G_{max} \subseteq \ldots \subseteq G_\ell \subseteq \ldots \subseteq G_2 \subseteq G_1 \subseteq G_0 = G$$

$\mathbf{F_\ell}$ is subforest of F induced by edges of level $\geq \ell$.

$$F_{max} \subseteq \ldots \subseteq F_\ell \subseteq \ldots \subseteq F_2 \subseteq F_1 \subseteq F_0 = F$$

Invariants

Recall: F_ℓ subforest of F induced by edges of level $\geq \ell$.

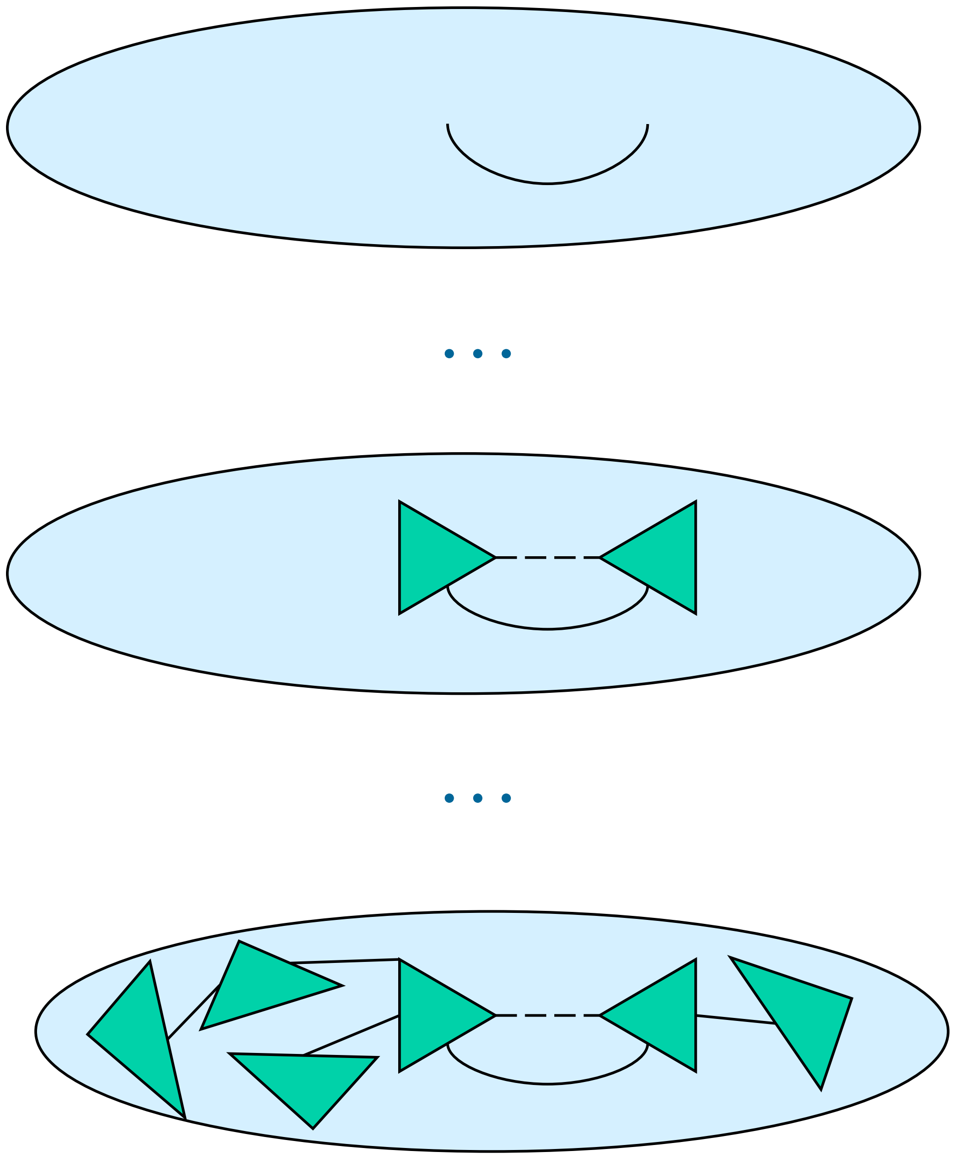
Will keep the following two invariants:

(Invariant 1) Each tree in F_ℓ (i.e., connected component in G_ℓ) has at most $n/2^\ell$ vertices

➔ At most (log n) levels

(Invariant 2) The forest F is a maximum (with respect to ℓ) spanning forest, that is if (v, w) is a non-tree edge of level ℓ, then v and w are connected (i.e., in the same tree) in F_ℓ

➔ If a tree edge at level ℓ is deleted, then a replacement edge (if there is one) must be of level $\leq \ell$



$$F_{\log n} \supseteq \cdots \supseteq F_\ell \supseteq \cdots \supseteq F_0 = F$$

Observations

Initially all edges at level 0 (both invariants satisfied)

Amortization argument: Levels of an edge can only increase, so we can have ≤ log n increases per edge

Intuition: When level of non-tree edge increased, it is because we discovered that its endpoints are close enough in F to fit in a smaller tree (higher level)

Increasing the level of a tree edge is always safe for Invariant 2 (F is a maximum spanning forest) but it may violate Invariant 1

Invariant 1

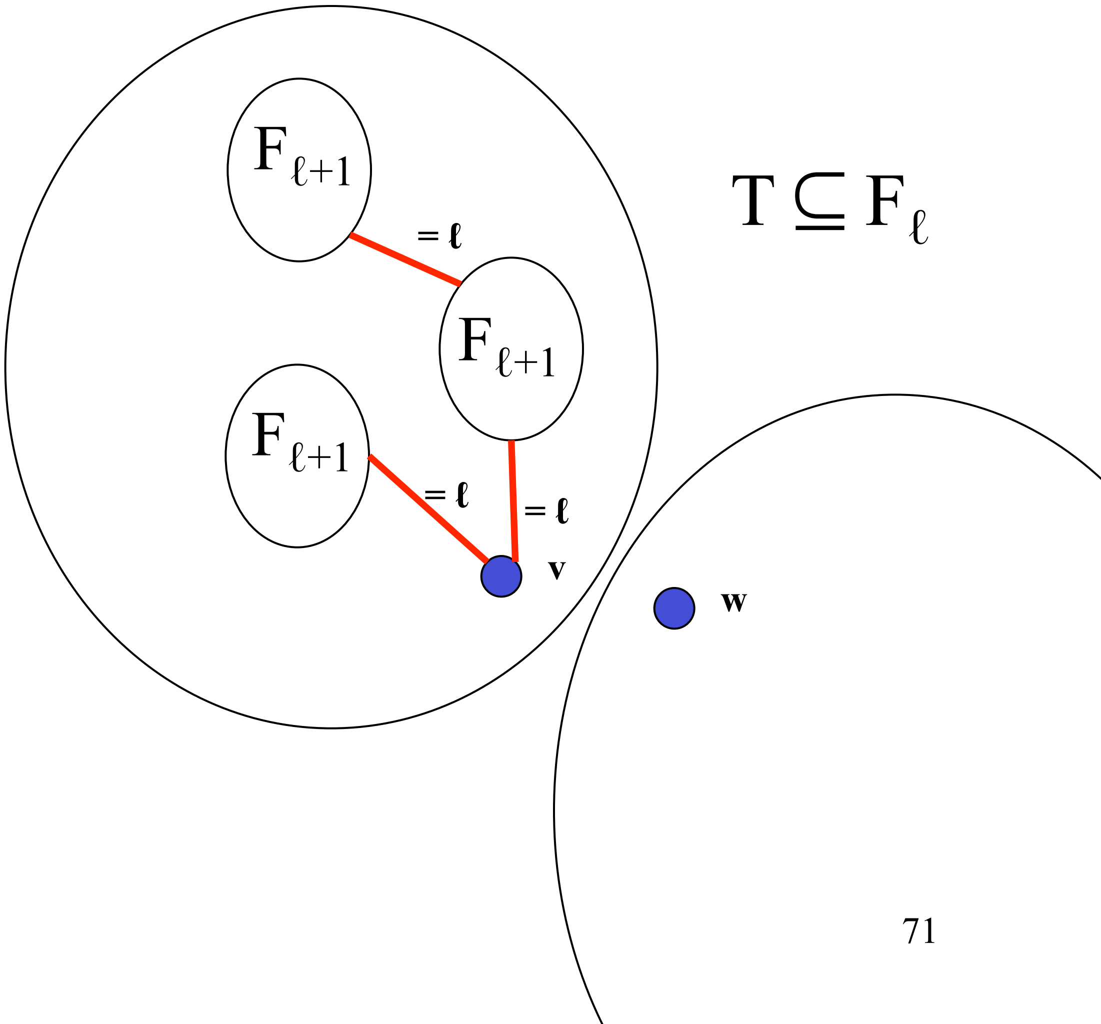
$|T| \leq n/2^{\ell}$

$|T_v| \leq |T_w|$

➔ $|T_v| \leq n/2^{\ell+1}$

We can afford to push all edges of T_v from level ℓ up to level $\ell + 1$ (while still preserving Invariant 1).

The replacement edge stays at level ℓ



Implementation

For each level ℓ:

- Maintain F_ℓ in a dynamic tree data structure.

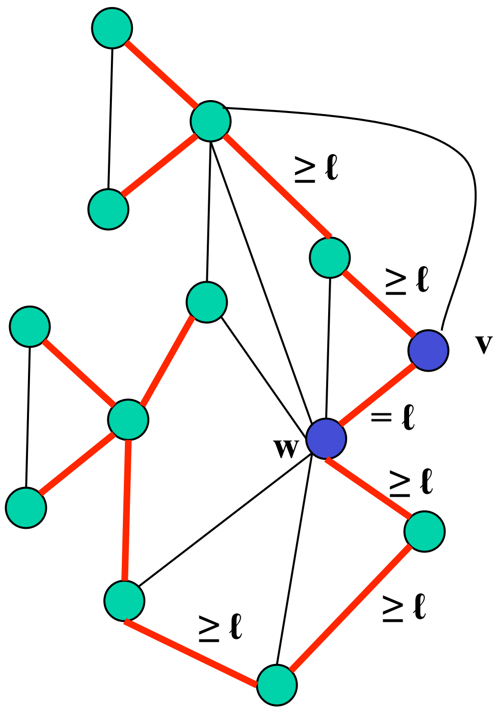
For each vertex v and each level ℓ:

- Maintain a list of incident tree edges and a list of incident non-tree edges at that level.
  (So each vertex has 2 lists per level, i.e., a total of 2 log n lists.)

Each vertex replicated in at most log n levels

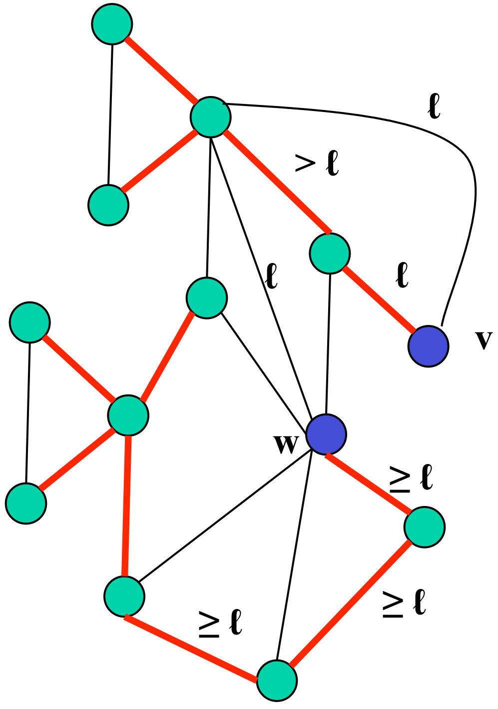
Thus, space usage will be $O(m + n \log n)$

Suppose a tree edge of level ℓ, say (v,w), is deleted. Then (v,w) belongs to some tree T of F_ℓ



Suppose a tree edge of level ℓ, say (v,w), is deleted. Then (v,w) belongs to some tree T of F_ℓ

If there is a replacement at level ℓ then it must be incident to one of the pieces of T



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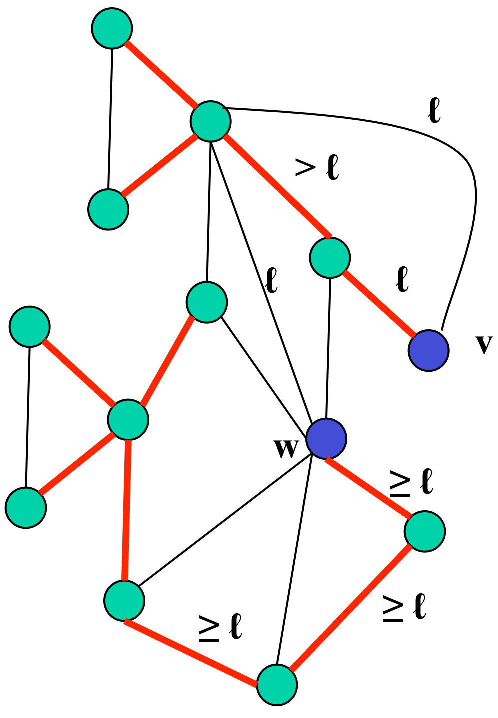
Let T_v and T_w be the pieces of T in F_ℓ containing respectively v and w after deleting edge (v,w). W.l.o.g. assume $|T_v| \leq |T_w|$.



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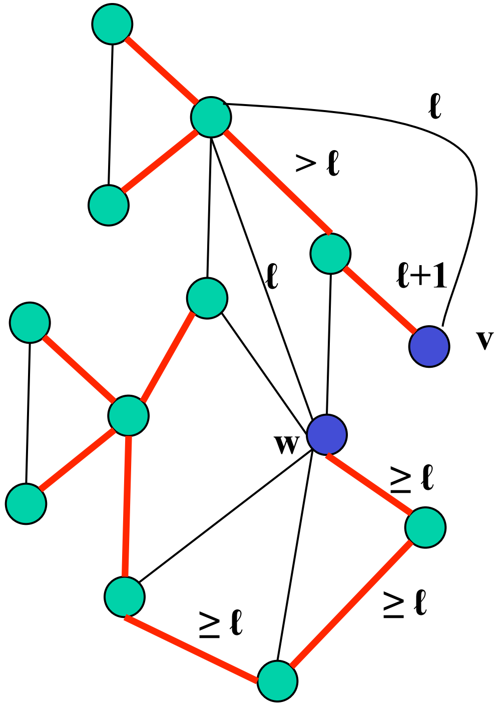
We increase to $\ell+1$ the edges of level ℓ in T_v



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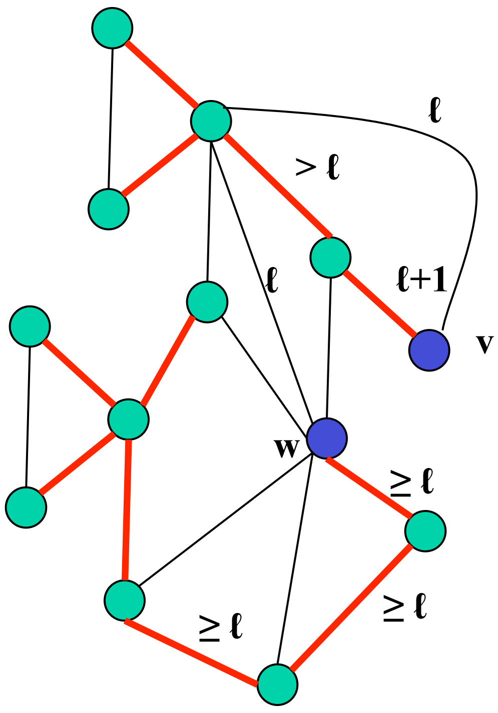
Next, we traverse all level ℓ non-tree edges incident to T_v to find a level-ℓ replacement edge.



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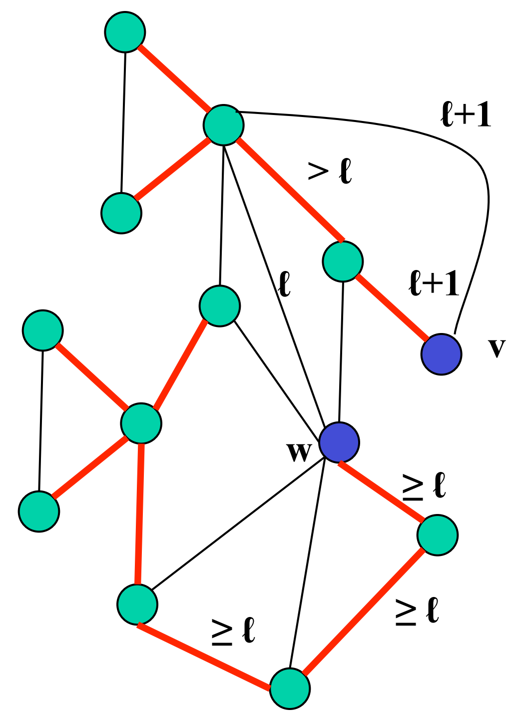
If a traversed edge is not a replacement we increase its level to $\ell+1$



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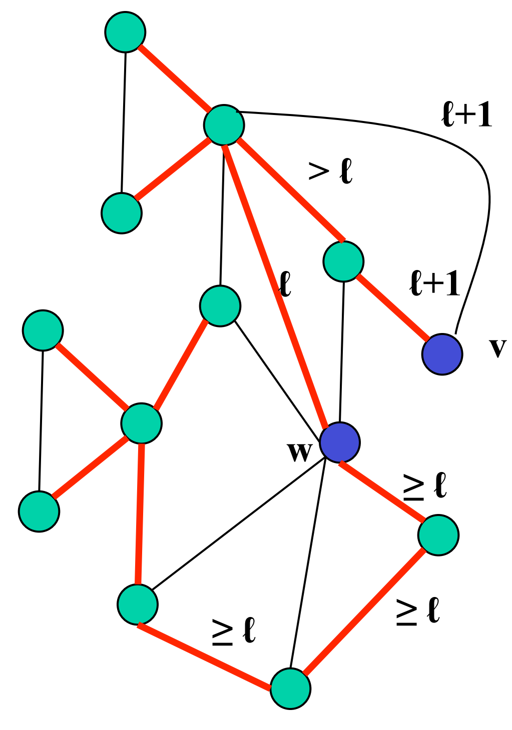
If there is a replacement edge at level ℓ, then we are done



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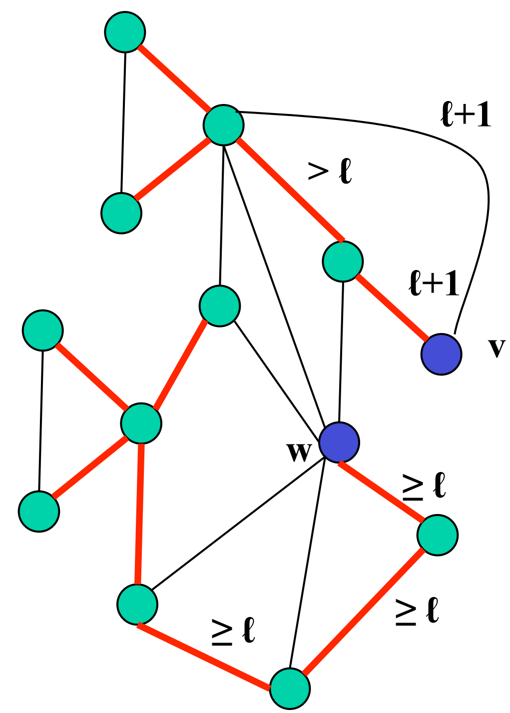
We increase to $\ell+1$ the edges of level ℓ in T_v

Next, we traverse all level ℓ non-tree edges incident to T_v to find a level-ℓ replacement edge.

If a traversed edge is not a replacement we increase its level to $\ell+1$

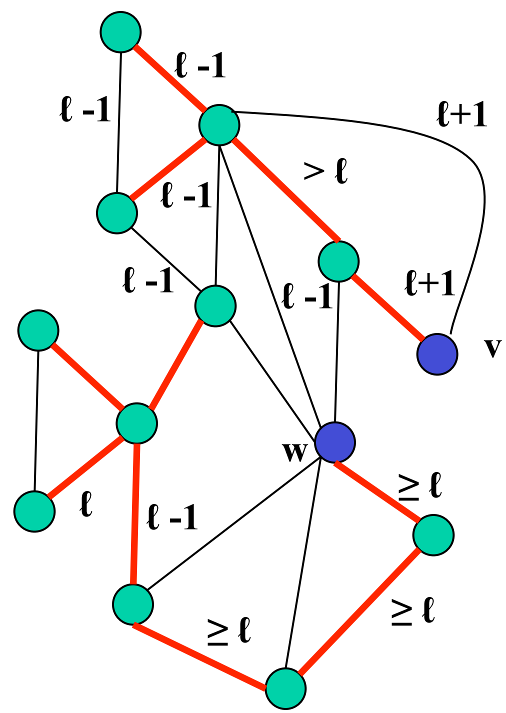
What if there is a no replacement edge at level ℓ?



If there is no replacement edge of level ℓ we look for replacement edges of level ℓ - 1

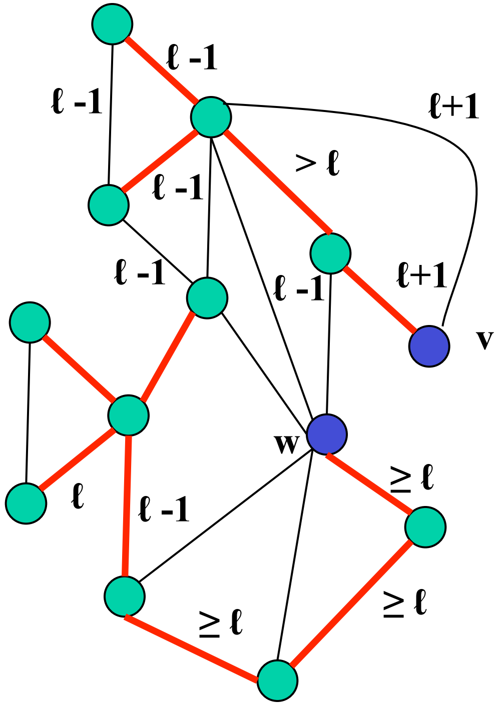
Let T_v and T_w be the trees in $F_{\ell-1}$ after deleting (v,w) containing v and w respectively

Assume $|T_v| \leq |T_w|$: then we increase the level of edges of level ℓ-1 in T_v to be ℓ and we start traversing the non-tree edges of level ℓ-1 incident to T_v



We keep going down like that level by level and either we find a replacement edge or we conclude that no replacement edge exists

As we go, we keep our invariants



Implementation

- We keep each forest $F_0 \subseteq F_1 \subseteq \ldots \subseteq F_{\log n}$ separately
- The non-tree edges of level ℓ are kept with the nodes of F_ℓ

Implementing the operations

connected(v,w) :

Check whether v and w are in the same tree of F_0

insert(v,w) :

If v and w are in different trees of F_0 add the edge to F_0 (i.e., at level 0). Otherwise, just add a non-tree edge of level 0 to v and w.

Both invariants are still satisfied.

Implementing the operations

delete(v,w):

Let ℓ be the level of edge (v,w).

• If (v,w) is a non-tree edge of level ℓ then simply delete it from v and w in $F_{\ell.}$

• Otherwise, delete (v,w) from the trees containing it in F_ℓ , $F_{\ell-1}$, … , F_0 and find a replacement edge as described before (at the highest possible level). If a replacement edge (x,y) is found at level $k \leq \ell$, then add (x,y) to F_k, F_{k-1}, … , F_0

Operations we need to do on the forests

For each ℓ, wish to maintain the forest F_ℓ together with all non-tree edges on level ℓ.

For any vertex v, wish to find the tree T_v in F_ℓ containing it

Want to be able to compute the size of T_v

Want to be able to find an edge of T_v on level ℓ, if one exists.

Want to be able to find a level ℓ non-tree edge incident to T_v, if any.

Operations we need to do on the forests

Trees in F_ℓ may be cut (when an edge is deleted) and linked (when a replacement edge is found, an edge is inserted or the level of a tree edge is increased).

Moreover, non-tree edges may be introduced and any edge may disappear on level ℓ (when the level of an edge is increased or when non-tree edges are inserted or deleted).

All this can be done in O(log n) time (by suitably augmenting ET-trees)

Analysis

- Query takes O(log n)
- Insert takes O(log n) time + charge the time to increase the level of the edge. Each level increase costs O(log n) (ET tree) so it $O(\log^2 n)$ total.
- Delete cuts and links O(log n) forests + level increases (charged to insert). Overall it takes $O(\log^2 n)$

(Main) History of the Problem

Update	Query	Type	Reference
$O(m^{1/2})$	$O(1)$	det/w-c	[Frederickson SICOMP'85]
$O(n^{1/2})$	$O(1)$	det/w-c	[Eppstein, Galil, I. & Nissenzweig JACM'97]
$O(\log^3 n)$	$O(\frac{\log n}{\log \log n})$	rand/amort	[Henzinger, King JACM'99]
$O(\log^2 n)$	$O(\frac{\log n}{\log \log n})$	rand/amort	[Henzinger, Thorup Rand. Struct. & Algs. '97]
$O(\log^2 n)$	$O(\frac{\log n}{\log \log n})$	det/amort	[Holm, de Lichtenberg & Thorup JACM'01]
$O(\log n (\log \log n))$	$O(\frac{\log n}{\log \log \log n})$	rand/amort	[Thorup STOC' 00]
$O(\frac{\log^2 n}{\log \log n})$	$O(\frac{\log n}{\log \log n})$	det/amort	[Wulff-Nilsen SODA'13]
$O(\log^5 n)$	$O(\frac{\log n}{\log \log n})$	rand/w-c	[Kapron, King & Mountjoy SODA'13]

Best (Published) Bounds

Update	Query	Type	Reference
$O(m^{1/2})$	$O(1)$	det/w-c	[Frederickson SICOMP'85]
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Best Bounds

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$O(\log n\,(\log\log n))$	$O\left(\frac{\log n}{\log\log\log n}\right)$	rand/amort	[Thorup STOC' 00]
$O\left(\frac{\log^2 n}{\log\log n}\right)$	$O\left(\frac{\log n}{\log\log n}\right)$	det/amort	[Wulff-Nilsen SODA'13]
$O(\log^4 n)$	$O\left(\frac{\log n}{\log\log n}\right)$	rand/w-c	[Gibb, Kapron, King & Thorn arXiv'15]

Lower Bounds

Update	Query	Type	Reference
$O((\frac{n}{\log n})^{1/2} \log \log n)$	$O(1)$	det/w-c	[Keilberg-Rasmussen, Kopelowitz, Pettie & Thorup, arXiv'15]
$O(\log n (\log \log n))$	$O(\frac{\log n}{\log \log \log n})$	rand/amort	[Thorup STOC' 00]
$O(\frac{\log^2 n}{\log \log n})$	$O(\frac{\log n}{\log \log n})$	det/amort	[Wulff-Nilsen SODA'13]
$O(\log^4 n)$	$O(\frac{\log n}{\log \log n})$	rand/w-c	[Gibb, Kapron, King & Thorn arXiv'15]

$O(x \log n)$ ⇒ $\Omega(\frac{\log n}{\log x})$

$\Omega(\frac{\log n}{\log x})$ ⇐ $O(x \log n)$

[Patrascu, Demaine SICOMP'06]

Open Problems: Close the Gaps

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$O(x \log n)$ ⇒ $\Omega\left(\frac{\log n}{\log x}\right)$

$\Omega\left(\frac{\log n}{\log x}\right)$ ⇐ $O(x \log n)$

[Patrascu, Demaine SICOMP'06]

Open Problems

- Improve best known bounds, and in particular:
- Deterministic algorithm with O(polylog n) update and query in the worst case?
- Randomized Las Vegas algorithm with O(polylog n) update and query in the worst case?
- Deterministic / randomized algorithm with O(log n) update and query?
- Deterministic / randomized algorithm with o(log n) update and O(polylog n) query?

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