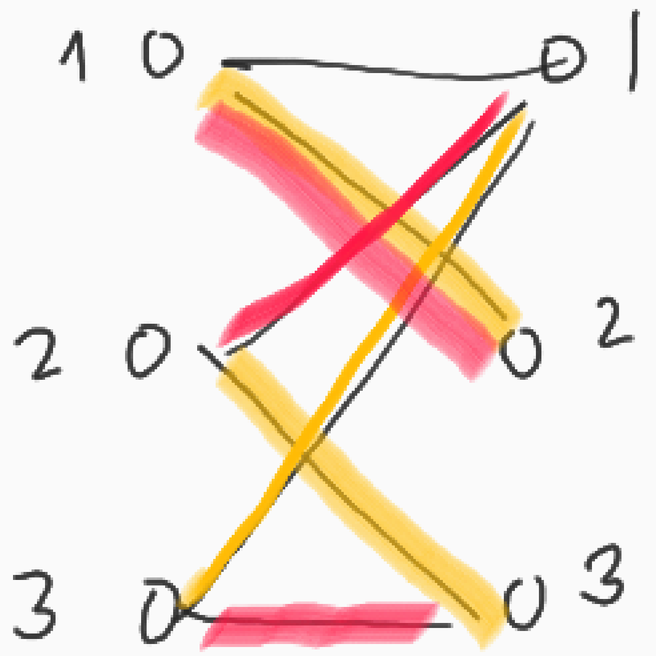
Lecture 9: Matchings using Matrix-Methods

Randomization + Algebra = Magic!

- Polynomial Identity Testing (PIT)
- Lovasz's Algorithm ←
- Red-Blue Matchings ←



$$\boxed{P(x_1, x_2 \ldots x_n) \stackrel{?}{=} 0 \quad ?}$$

$$\mathcal{E} = \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & 0 & x_{23} \\ x_{31} & 0 & x_{33} \end{pmatrix}$$

$$\det(\mathcal{E}) = -\underbrace{x_{12}(x_{21}x_{33}}- x_{23}x_{31})$$

$$=$$

$$\mathcal{E}_{ij} = \begin{cases} x_{ij} & ij \in E \\ 0 & \text{if } ij \notin E \end{cases}$$

Polynomial Id Testing

$P(x) = 5x^3 - 3x + 9$

$P(x_1, x_2) = \underbrace{3x_1^{2} x_2^{0}}_{\text{monomial degree 3}} + 7x_2^{5} + 19x_1 - \frac{17}{2}x_3 + 9$ ← degree $= 5$

Working over field $\mathbb{F}$

- $\mathbb{Q}$
- $\mathbb{F}_p$

- all arithmetic ops take $O(1)$ time

  ops in $\mathbb{F}_p$ can be done in time $\text{poly}(\log p)$

Few-roots lemma

(non zero)

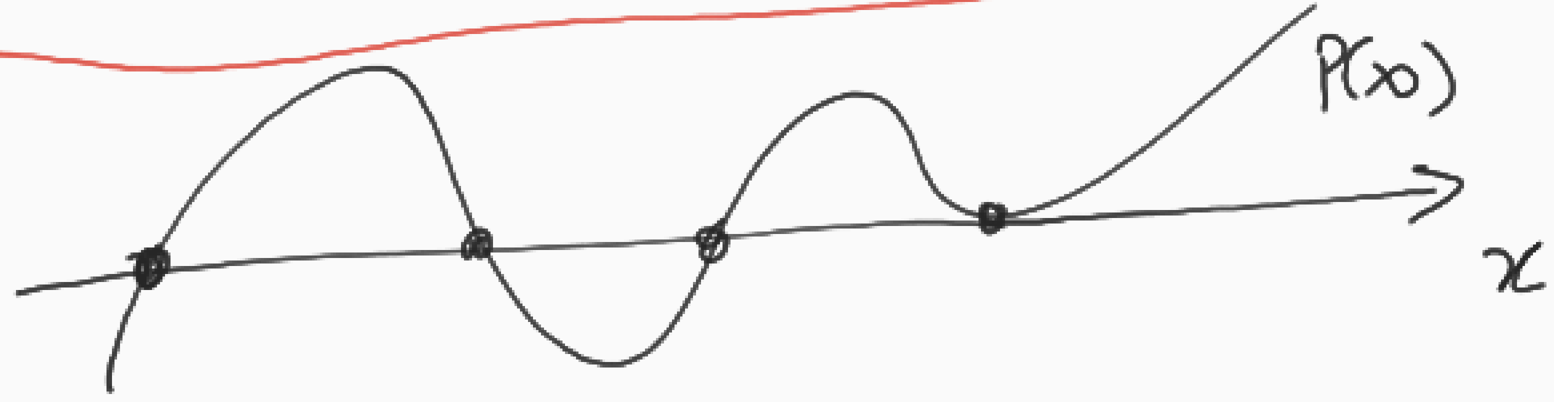
r s.t. $P(r) = 0$.

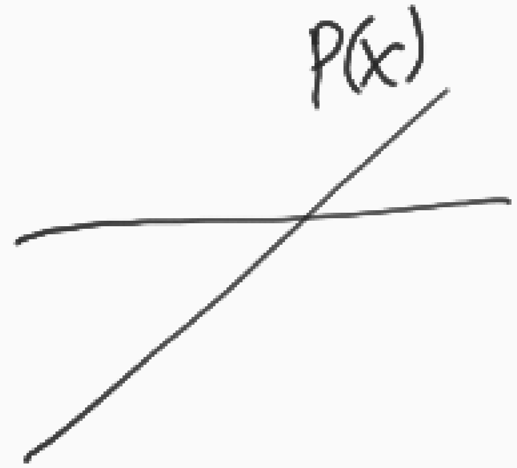
- $\boxed{\text{Polynomial (univariate) of degree} \le d \text{ has} \le d \text{ roots}}$ (over any $\mathbb{F}$)

  $P(x)$

$P(x) = 5$

$P(x) = 0$





- $P(x,y) = (x-2)(y-5)$ $\quad$ $(2, ?)$, $(?, 5)$

  infinitely many roots

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Thm [Schwartz– Zippel Thm]

Take any non zero poly $P(x_1, x_2 \dots x_n)$ with degree $\leq d$. Choose values $R_1, R_2 \dots R_n$ for each R_i uniformly at random, independently from S.

then $\Pr[P(R_1, R_2 \dots R_n) = 0] \leq \frac{d}{|S|}$

$S \subseteq \mathbb{F}$

#(elements in set S^n that are roots of P)

$\leq \underbrace{\frac{d}{|S|}}\cdot\underbrace{|S|^n}$

Pf. Induction on n, # vars. $n = 1$ univariate

Else. $P(x_1 \dots x_n) = x_1^{\alpha}\,\boxed{Q(x_2 \dots x_n)} + R(x_1 x_2 \dots x_n)$

$\Pr(P=0) = \cancel{\Pr(P=0 \mid Q=0)}^{\leq 1}\,\underbrace{\Pr(Q=0)}_{\frac{d-\alpha}{|S|}} + \underbrace{\Pr(P=0 \mid Q\neq 0)}_{\leq \frac{\alpha}{|S|}}\,\cancel{\Pr(Q\neq 0)}^{\leq 1}$

$\frac{d}{|S|}$

$x_1^5 x_2 x_7 + x_1^3 x_3^2 + x_1^4 \dots$

Algo [Lovász]

① Algo for Perfect Matching Detection

Bipartite graphs.

$$\det(\mathcal{E}) = \sum_{\pi} (-1)^{\mathrm{sign}(\pi)} \underbrace{\prod_{i=1}^{n} \mathcal{E}_{i\pi(i)}}_{\neq 0 \text{ iff } \pi \text{ is a matching}}$$

Algo: $\boxed{\text{Set } |S| = n^6 \cdot 100}$

Choose x_{ij} unif@random, indep from S.

$\mathcal{E} \to \tilde{\mathcal{E}}$ (with random vals)

compute $\det(\tilde{\mathcal{E}})$ ← $O(n^3)$ time Gauss. Elim → $O(n^\omega)$

DetectPM(G) → YES ⇒ always correct

DetectPM(G) → NO ⇒ maybe wrong.

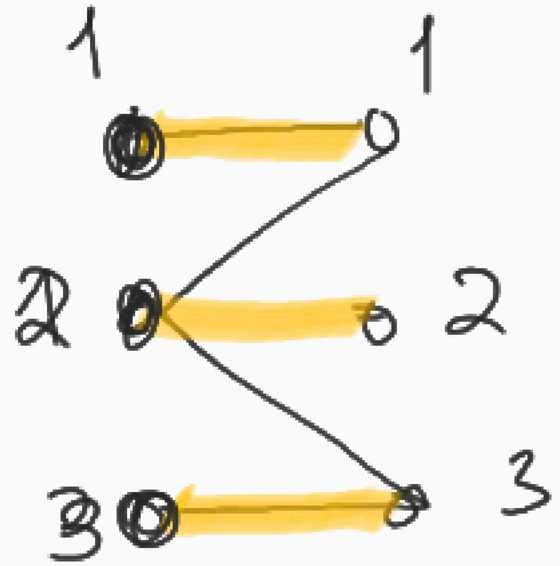
but $\Pr(\text{error}) \leq \frac{1}{\mathrm{poly}(n)}$ ← $100n^5$

Matrix

$\mathcal{E}$ st. $\det(\mathcal{E})$ is

① a degree-n poly in $\leq n^2$ vars

② non zero iff G has a PM



Non Bip Graphs

Tutte Matrix: $\mathbb{T} \in \mathbb{R}^{n \times n}$

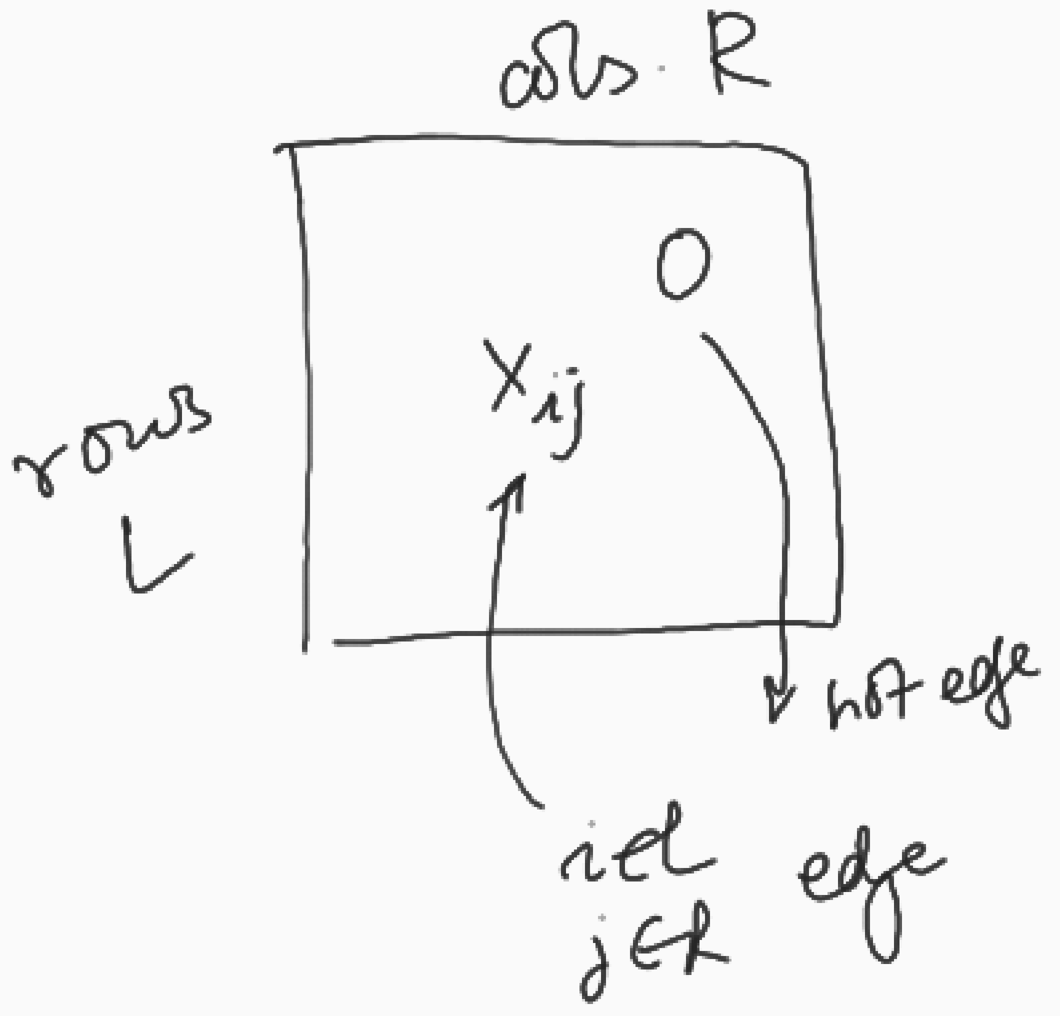
$|V| = n$

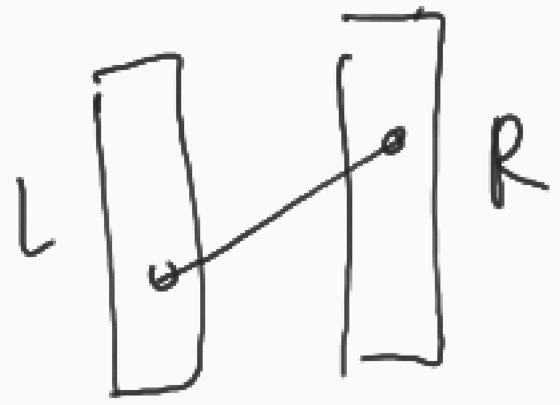
$V = \{1, 2 \, .. \, n\}$

$$\mathbb{T}_{ij} = \begin{cases} 0 & \text{if } ij \notin E \\ x_{ij} & \text{if } i<j \;\; ij \in E \\ -x_{ji} & \text{if } i>j \;\; ij \in E \end{cases}$$

skew symmetric

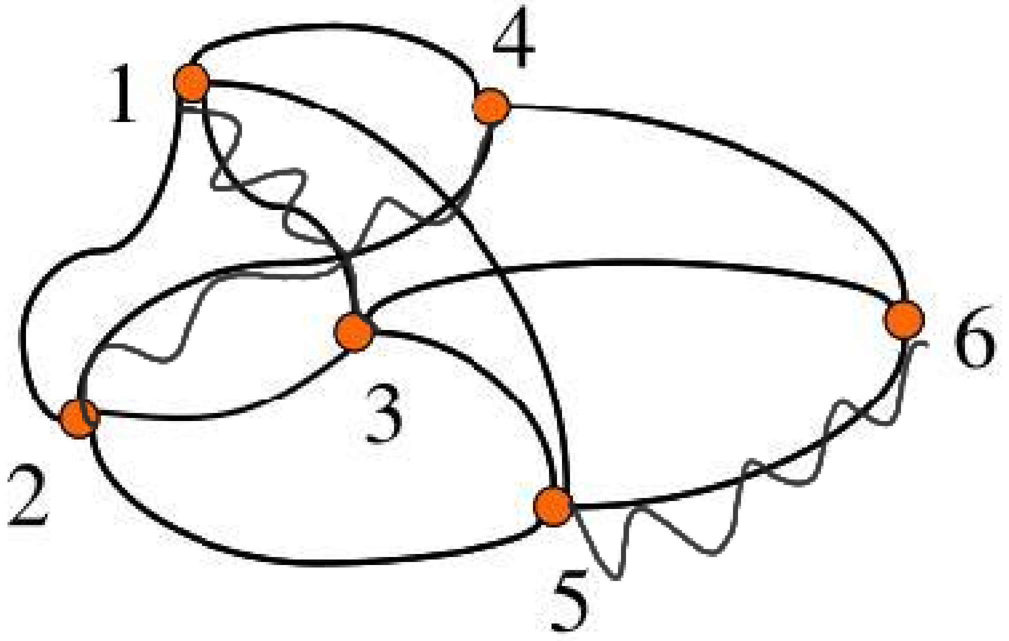
$$(\mathbb{T})^T = -\mathbb{T}$$





Tutte's matrix $(\pm)\, x_{13}^2\, x_{24}^2\, x_{56}^2$

(Skew-symmetric symbolic adjacency matrix)



$$\begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} & 0 \\ -x_{12} & 0 & x_{23} & x_{24} & x_{25} & 0 \\ -x_{13} & -x_{23} & 0 & 0 & x_{35} & x_{36} \\ -x_{14} & -x_{24} & 0 & 0 & 0 & x_{46} \\ -x_{15} & -x_{25} & -x_{35} & 0 & 0 & x_{56} \\ 0 & 0 & -x_{36} & -x_{46} & -x_{56} & 0 \end{pmatrix}$$

$$a_{ij} = \begin{cases} x_{ij} & \text{if } \{i,j\} \in E \text{ and } i < j, \\ -x_{ji} & \text{if } \{i,j\} \in E \text{ and } i > j, \\ 0 & \text{otherwise} \end{cases} \qquad A^T = -A$$

Thm [Tutte 4?] $\det(A) \neq 0 \iff A$ has a PM.

Q: Given algo to ~~test~~ $\underbrace{\text{detect PM}}$, $\underbrace{\text{find PM}}$.

$\underset{\sim}{A}$ $\quad$ $\boxed{\text{decision version}}$ $\quad$ $\boxed{\text{search version}}$

$\underline{\underline{T(n) \leq n^{\omega}}}$

A:

Test(G)

$\downarrow$ Yes

Test(G − e)

YES: $G \leftarrow G - e$

No. $\rightarrow$ G − {endpts of e}

$O(m \cdot T(n))$

Exercise:

$O(n \log n \cdot T(n))$ time.

Worldsearch: $O(n^{\omega})$ find PM

$O(n \cdot n^{\omega}) \leftarrow$ notes

Red Blue PM problem (Bipartite)

$G = (L, R, E)$ edges $\to$ red E_R

$\to$ blue E_B

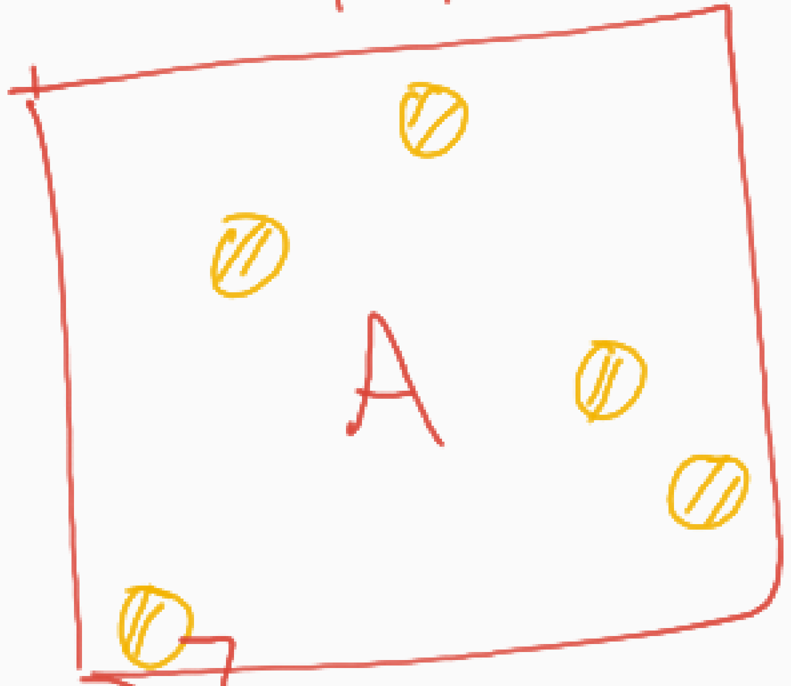
Given k, Find a PM in G that has exactly k red edges. $\leftarrow$ solve decision version

① Sps ≤ 1 PM with k red edges.

$$A_{ij} = \begin{cases} 0 & \text{if } ij \notin E \\ 1 & \text{if } ij \in E_B \\ y & \text{if } ij \in E_R \end{cases}$$

$|R| = n$

$n = |L|$



$$\det(A) = \sum_{\text{Matchings } M} (-1)^{\text{sign}(M)} \, y^{\#\text{red}(M)} = \left[\sum_{j \neq k} (\;) \pm y^k \right]$$

degree $\leq n$

$\underline{\det(A)}$ = poly in y, contains a term for $\underline{\underline{y^k}}$

iff $\exists$ a PM with k red edges.

$P(y)$

interpolate by evaluating it at $y = 1, 2, \ldots, n+1$

$P(1), P(2) \ldots P(n+1)$

reconstruct $P(y)$

Multiple PMs with k reds?

1. Choose random values for x_{ij} s
   $\Rightarrow$ poly in y.
2. Interpolate to reconstruct

$$A_{ij} = \begin{cases} 0 & \text{if } ij \notin E \\ y \cdot x_{ij} & \text{if } ij \in E_R \\ 1 \cdot x_{ij} & \text{if } ij \in E_B. \end{cases}$$

PIT $\in$ det poly time

$\Rightarrow$ circuit lower bounds