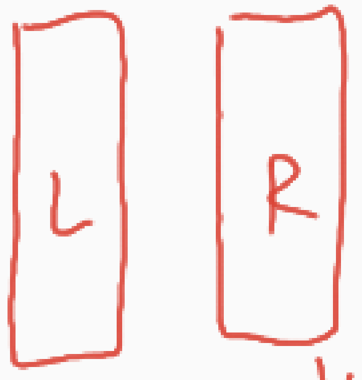
Lecture 8: Max/Min Weight Perfect Matchings (LP)



$|L| = |R| = n/2$

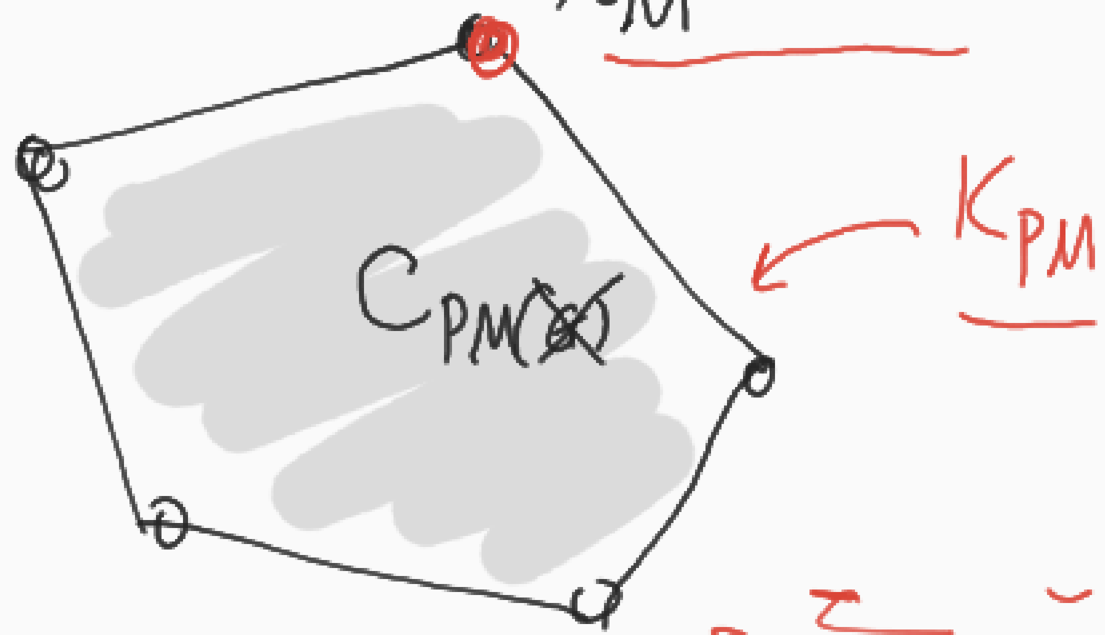
- **Perfect Matching Polytope**: convex hull of all perfect matchings (C_{PM})
- Write K_{PM} linear program, $\boxed{\text{show } K_{PM} = C_{PM} \text{ (for bipartite graphs)}}$

  $\Rightarrow$ can solve min/max weight PM in poly time.

$|E| = m$

$X_M \in \{0,1\}^m$

- 3 proofs (3 algorithms)
  - ✓ extreme point
  - basic feasible solution
  - vertex

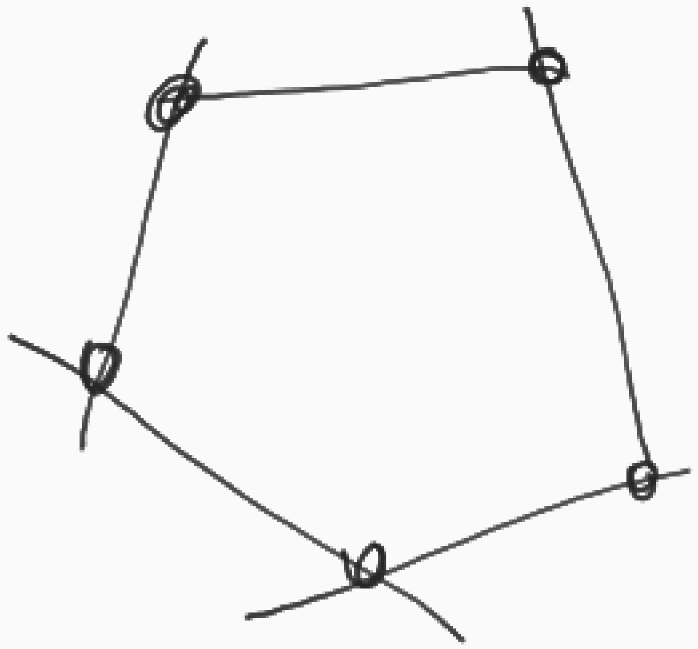


$\max\{ w \cdot x \mid x \in K_{PM} \}$

$$K_{PM} = \left\{ x \in \mathbb{R}^{|E|} \;\middle|\; \sum_{e \in \partial v} x_e = 1 \quad \forall v \in V \right.$$
$$\left. x_e \geq 0 \quad \forall e \in E \right\}$$

polytope.

<u>Claim</u> "every "corner" of K_{PM} is χ_M ← perfect matchig



- bfs.
- vertex

x^* bfs of K_{PM} is a matching $\chi_M \in \{0,1\}^m$

- feasible sat'y all constraints
- m tight constraints (lin ind)

↑ basis

$$x(A) = \sum_{e \in A} x_e$$

$$x(\partial v) = \sum_{e \in \partial v} x_e$$

$$\boxed{x(\partial v_1) \geq 1}$$

$$x(\partial v_2) = 1$$

$$\boxed{\vdots}$$

$$\boxed{x(\partial v_n) = 1}$$

$$\boxed{x_{e_1} \geq 0}$$

$$x_{e_2} \geq 0$$

$$\boxed{\vdots}$$

$$\boxed{x_{e_m} \geq 0}$$

m tight LI constraints at x^*

$$\Rightarrow \boxed{Bx^* = b}$$

$m \times m$ submatrix

$\det(B) \neq 0$

0, 1

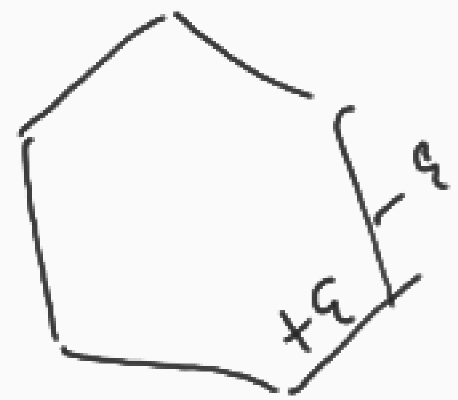
$$x^* = B^{-1} b$$

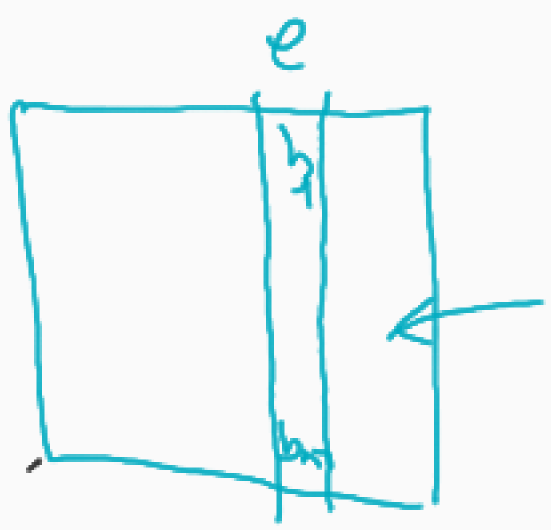
$$Ax \geq 1$$
$$x \geq 0$$

Claim: $x^*_e \in \mathbb{Z}$ (+ feasible $\Rightarrow \{0,1\}$)

$\mathbb{Z}^{m \times m}$

$$x^*_e = \frac{\det(B[b]_e)}{\det(B)} \quad \begin{matrix} \in \mathbb{Z} \\ \in \{\pm 1\} \end{matrix}$$



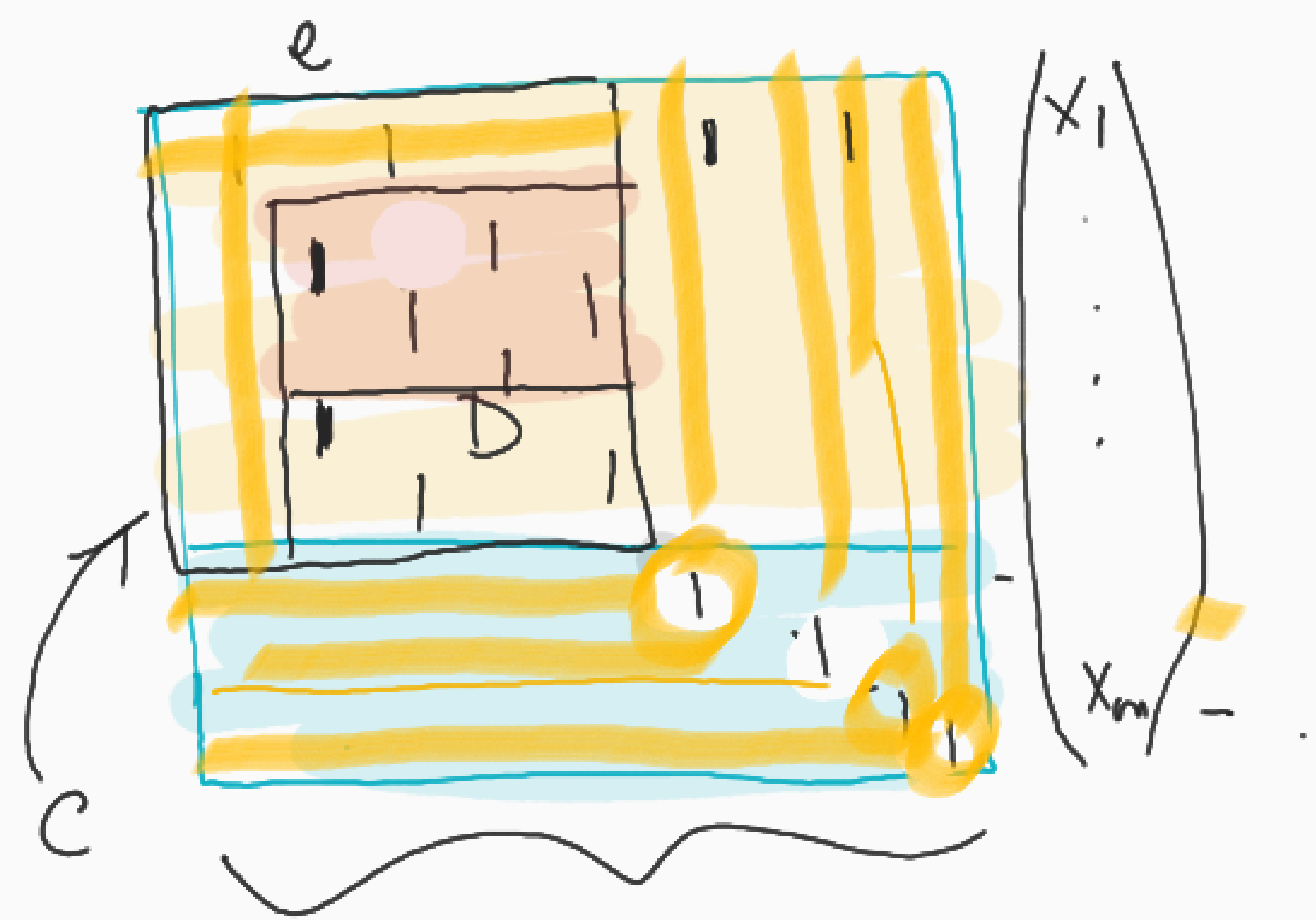


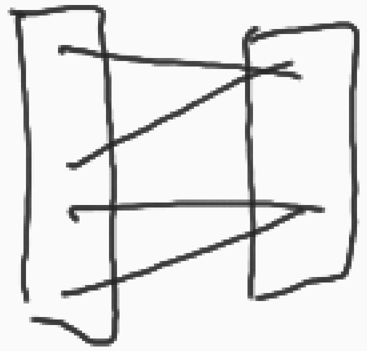
Subclaim: $\det(B) \in \{\pm 1, 0\}$

Pf: $\det(B) = \pm \det(C)$

expand using columns having single 1.

$\det(C) = \pm \det(D)$

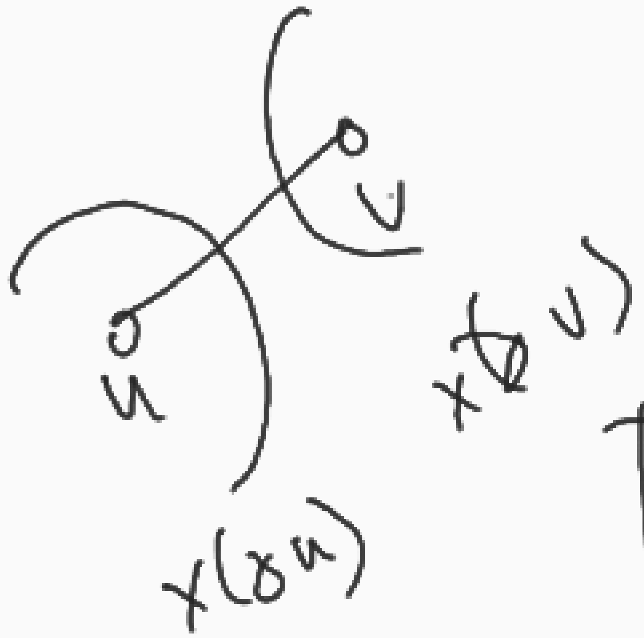




sum of rows

= sum of rows

$\det(D) = 0$ unless $D = 0 \times 0$ matrix



$x_3 = 0$

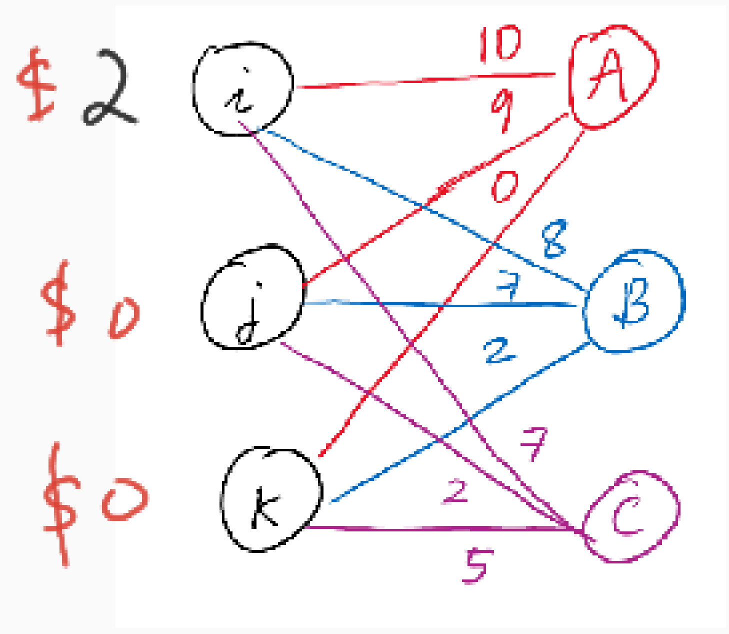
$x_{14} = 0$

$x_m = 0$

Lau, Ravi, Singh
Iterative Rounding

Max Weight PMs (buyers | items)

n items $\quad$ n buyers



$$\max_{\text{PM } M} \sum_{\text{buyers } b} v_{bM(b)}$$

$M(b)$: item assigned to b. by M.

$v_{bi} \in \mathbb{Z}$

Prices $p_i \quad \forall i \in I \qquad p = (p_1, p_2 \ldots p_n)$

utility of b for item i @ price $\bar{p}$

$$u_{bi}(p) = v_{bi} - p_i$$

max-utility of b @ $\bar{p}$

$$u_b(p) = \max_i (v_{bi} - p_i)$$

item i is preferred by b

if $u_{bi}(p) = u_b(p)$

Algo: . sets init prices $p_i \leftarrow 0$.

. draw the preferred graph $G(p)$

. if $G(p)$ has a PM, return it

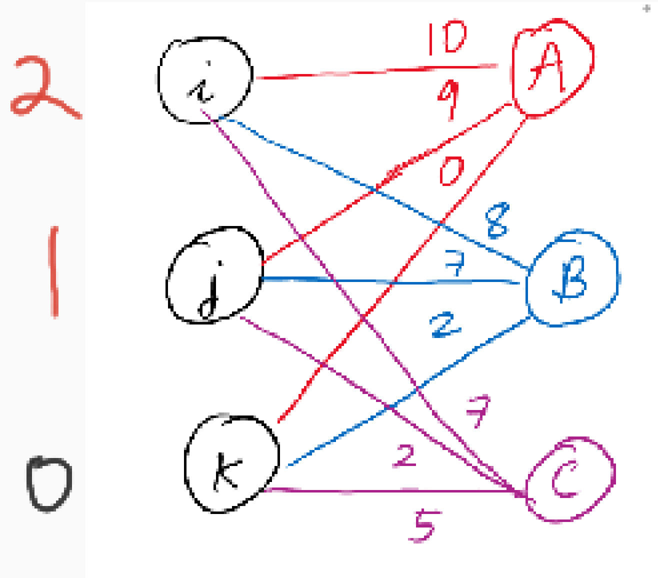
else find a hall set S st $|N(S)| < |S|$

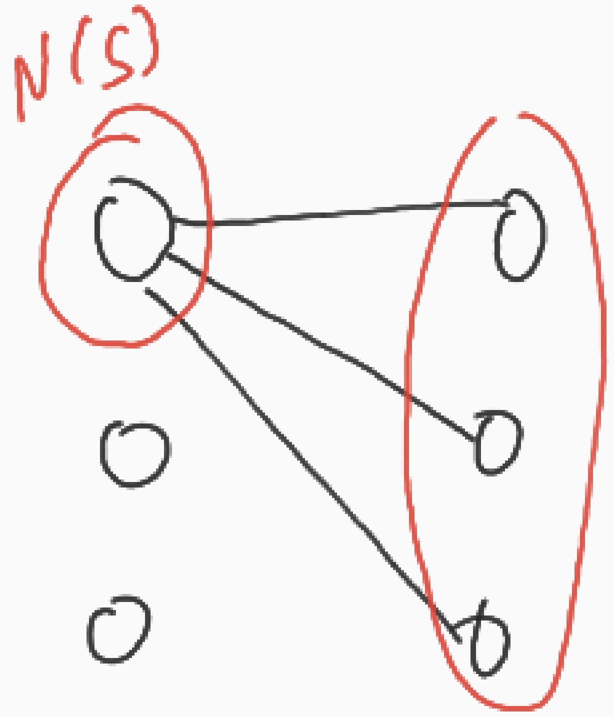
via Konig

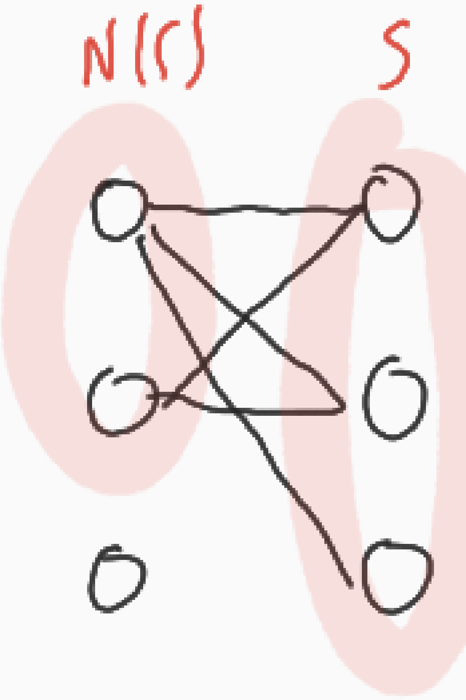
overdemanded items

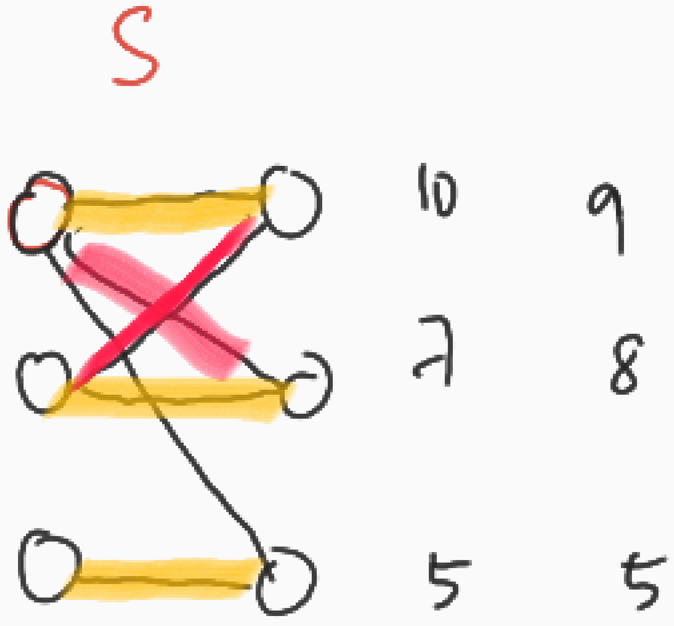
raise prices in $N(S)$ by \$1

Harold Kuhn.









Claim 1: Algo stops

Claim 2: when stops, M is a max-value PM.

$$\max \sum_i V_{bi} x_{bi}$$

primal

$$\text{st} \quad \sum_i x_{bi} = 1 \quad \forall b \qquad u_b$$

$$\sum_b x_{bi} = 1 \quad \forall i \qquad p_i$$

$$x_{bi} \geq 0$$

$$\min \sum_i p_i + \sum_b u_b$$

$$p_i + u_b \geq V_{bi} \quad \forall\, b\, i$$

$$\boxed{\underline{\text{Claim}}:\ u_b \stackrel{\text{def}}{=} \max_i (V_{bi} - p_i)}$$

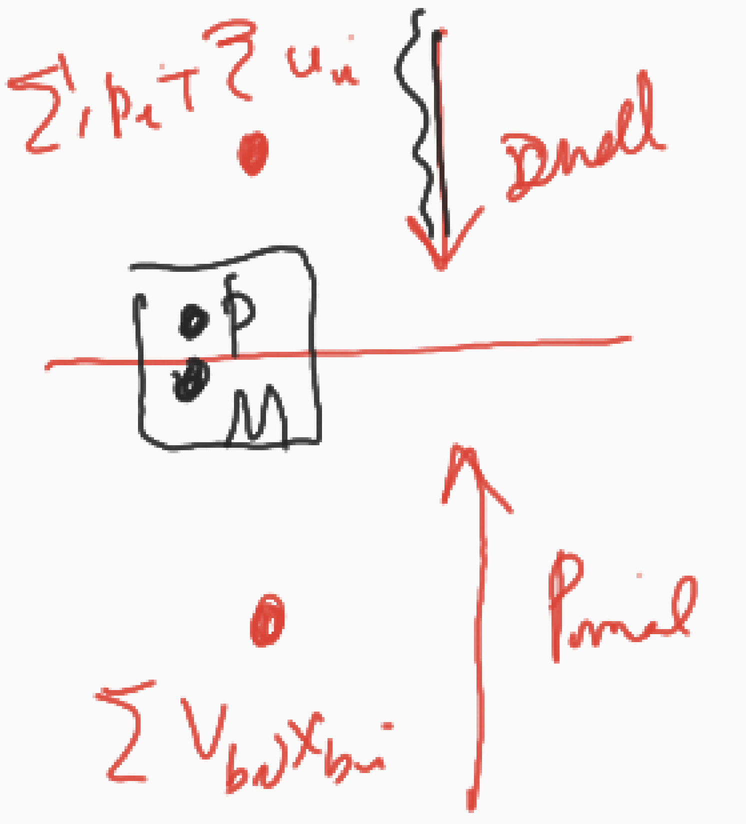
$$\min_p \sum_i p_i + \sum_b u_b(p)$$

<u>Weak duality</u>

$\forall$ primal sol^n x

$\forall$ dual sol^n (p, u)

primal value $\leq$ dual value



<u>Strg duality</u>

OPT primal = OPT dual

<u>Claim 2</u>: if p prices $\Rightarrow$ $G(p)$ has perf match M

$\Rightarrow$ $\underline{\underline{M}}$ is optimal (max wt)

$$\boxed{x^* = X_M}$$

$x^*_e = 1$ $e \in M$

0 if not

<u>Pf</u>: $\displaystyle val(M) = \sum_{bi} V_{bi} x^*_{bi}$

$$= \sum_{bi} (V_{bi} - p_i) x^*_{bi} + \underbrace{\sum_{bi} p_i x^*_{bi}}$$

$$= \sum_b u_b(p) + \sum_i p_i$$

$= \text{dual value}$

Claim 1: Algo stops. (integer values)

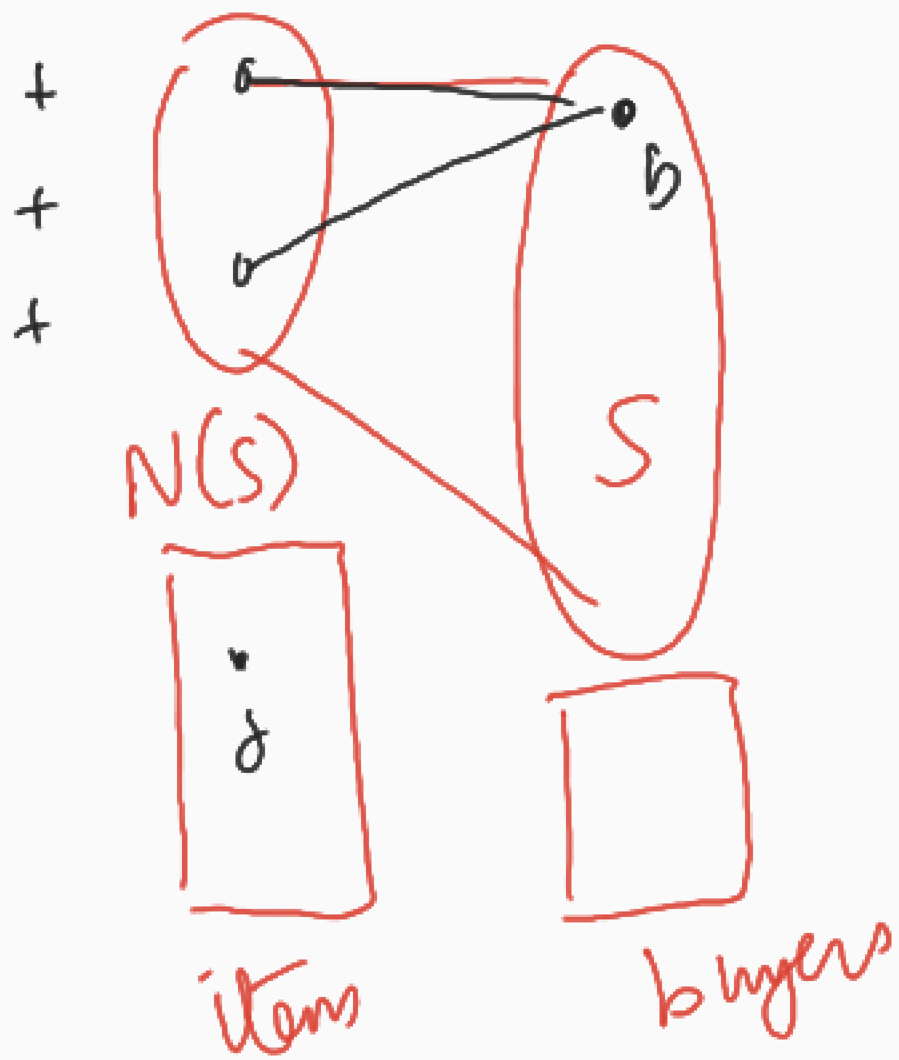
Dual value (over time)

- $\Phi = \underbrace{\sum_i p_i}_{\text{inc by } |N(S)|} + \underbrace{\sum_b u_b(p)}_{\text{dec by } |S|}$
- @ each step, $\Phi \downarrow$ at least 1.
-

$\Rightarrow \Phi \downarrow \quad |S| - |N(S)| \geq 1$

$\Phi_{init} = \sum_b \left(\max_i v_{bi} \right)$

Φ_{final} = OPTimal value of PM



① Primal – Dual

② English or ascending price auctions

– sell multiple items

– auction design

③ max Wt PM algo.