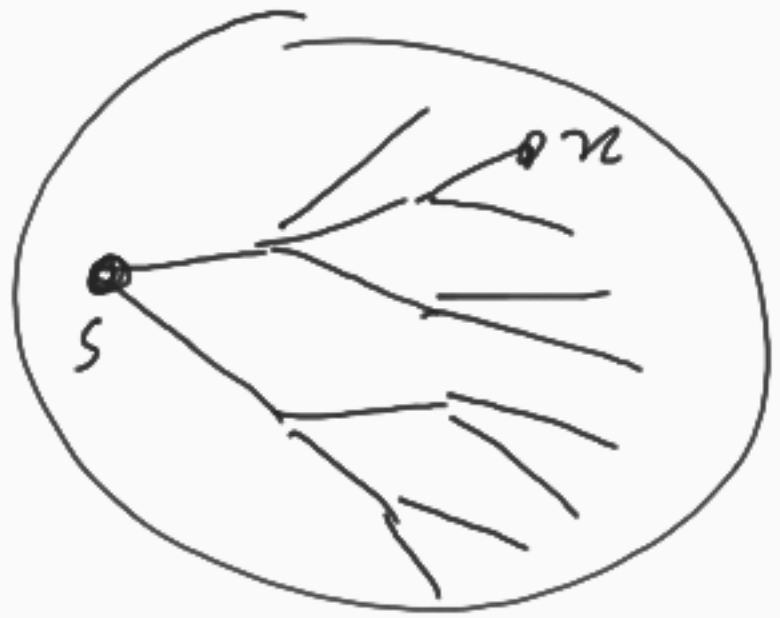
Lecture #5: Distance Preserving and Low-Stretch Trees

$G = (V, E)$ undirected $w_e \geq 0 \Rightarrow d(x,y)$

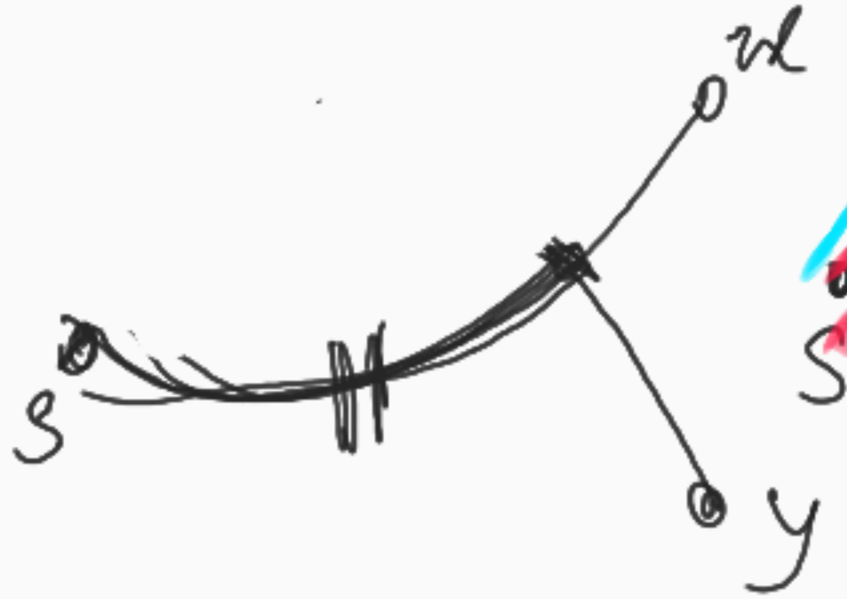
record!

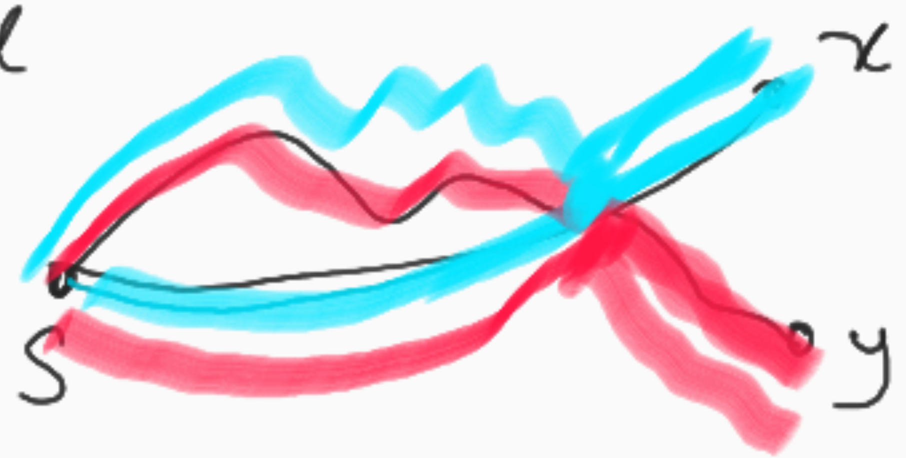
Shortest Path Tree SSSP

1. SSSP trees $\Rightarrow$ APSP trees?
2. Application $\leftarrow$
3. Results
4. Low Diameter Decompositions (LDDs)
5. Low Stretch Trees (LSTs)



$d_T(s,x)$
$= d_G(s,x)$





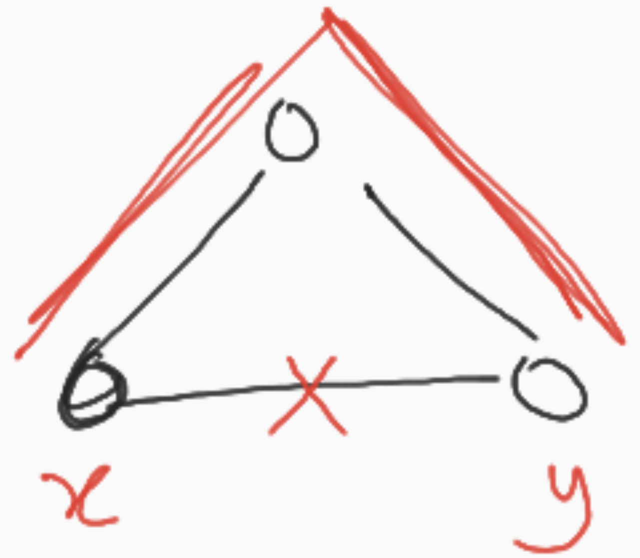
APSP tree?

want spanning tree T of G

st $d_T(x,y) \overset{?}{=} d_G(x,y)$?

shortest path distances in T

No :(



$\boxed{\alpha \geq 1}$

T st

① $d_T(x,y) \geq d_G(x,y)$

② $d_T(x,y) \leq \alpha \cdot d_G(x,y)$

stretch

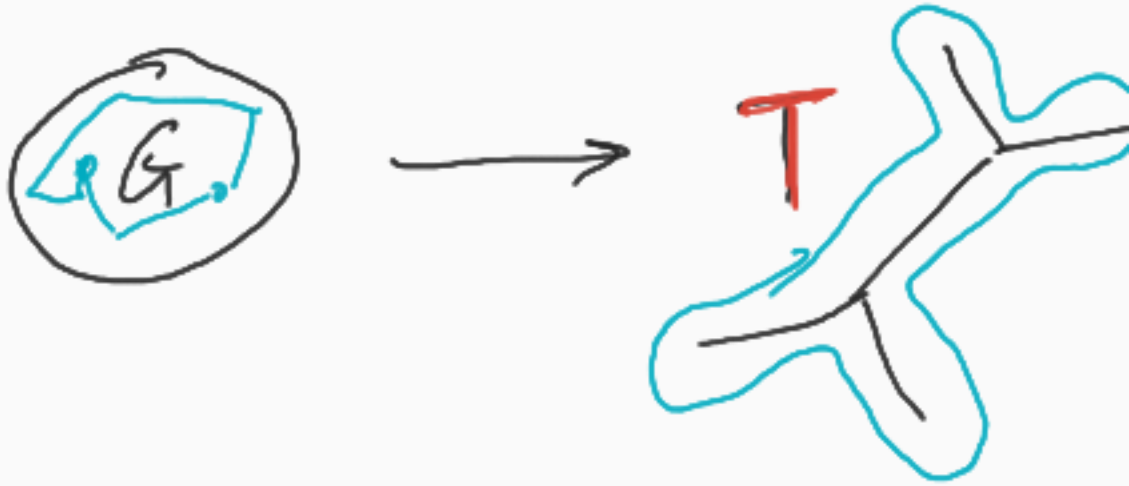
Traveling Salesperson Problem (TSP)

G   tour visits all n ~~cities~~ nodes and has min length.

TSP is hard on gen graphs.   TSP easy on trees.

Sps had APSP tree T with "stretch" α

$\Rightarrow$ α apx sol^n to TSP on general graph



π_T

run π_T on G

Claim: π_T is a α-apx sol^n

Pf: Sps π^* OPTimum in G

$$OPT_G = \sum_{i=1}^{n} d_G(\pi^*_i, \pi^*_{i+1})$$

Sps consider π^* as sol^n on T

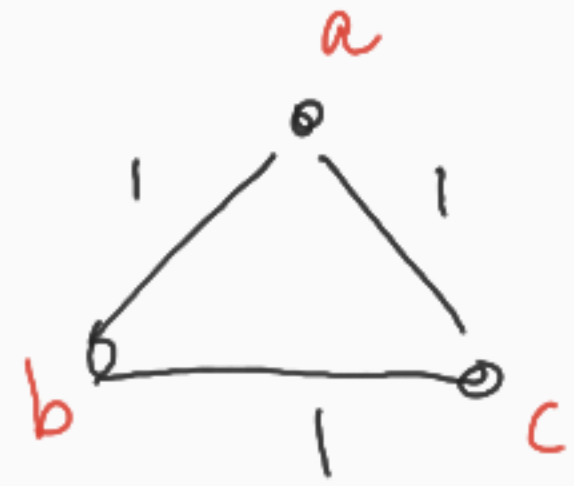
① $$\mathop{E}_{T}\left[cost(\pi^*, T)\right] = E\sum_i d_T(\pi^*_i, \pi^*_{i+1}) \overset{②}{\leq} \alpha \underbrace{\sum_i d_G(\pi^*_i, \pi^*_{i+1})}_{OPT}$$

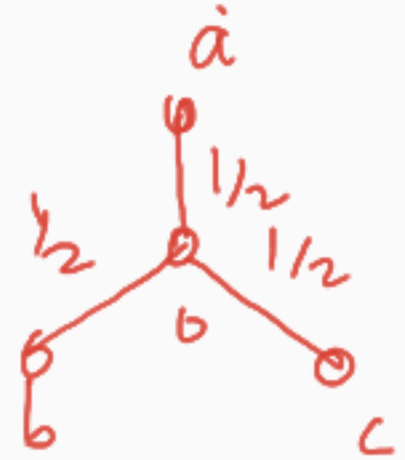
② $$cost(\pi_T, T) \leq cost(\pi^*, T)$$

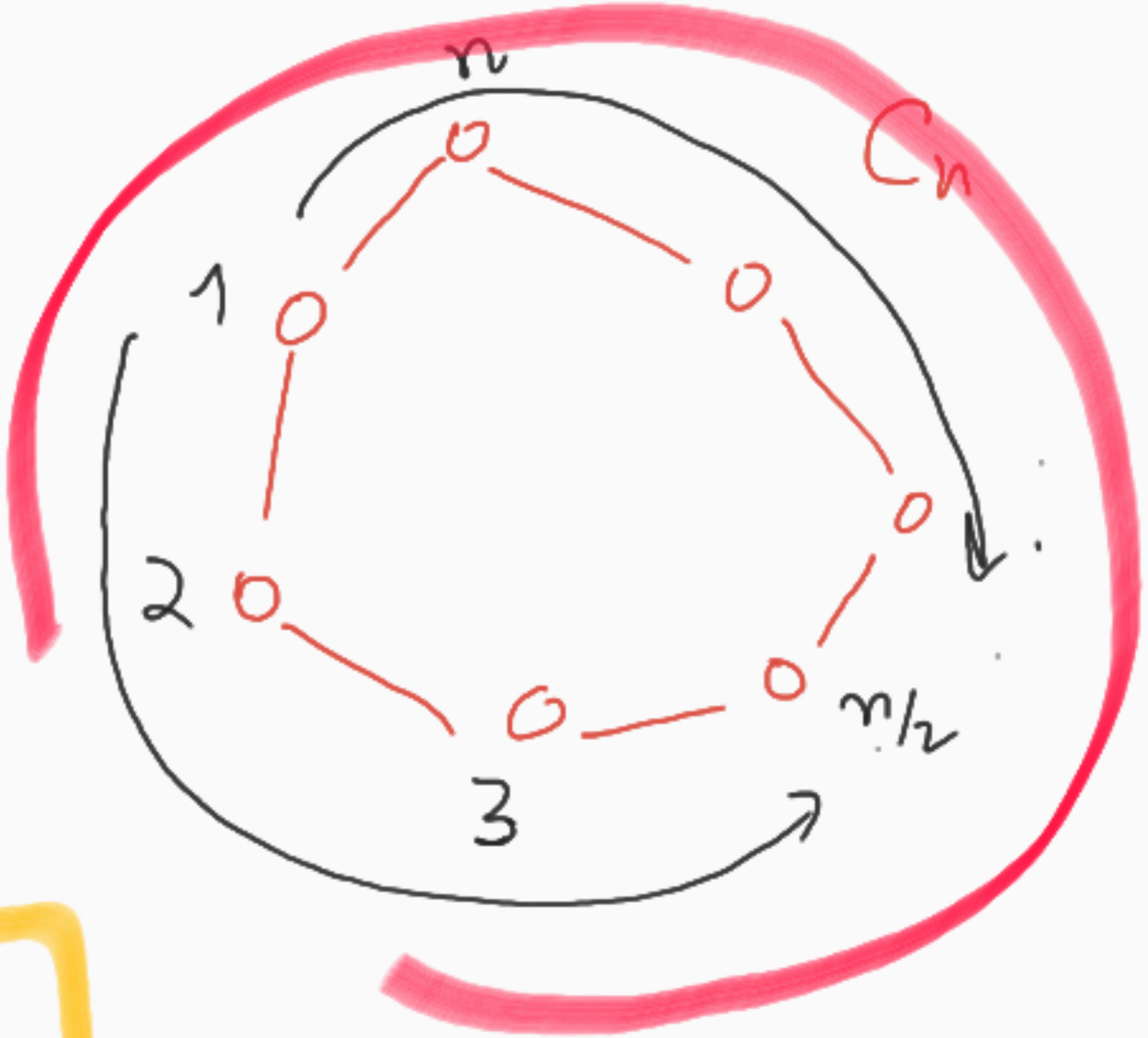
③ $$\boxed{ALG} = E\left[cost(\pi_T, G)\right] = E\left[\sum_i d_G(\pi^T_i, \pi^T_{i+1})\right] \overset{①}{\leq} \sum d_T(\pi^T_i, \pi^T_{i+1})$$

$= cost(\pi_T, T)$

$$E[ALG] \leq \alpha\, OPT$$







$$d_{ij} = \min \{ |i-j|, |n-(i-j)| \}$$

Fact: every subtree has stretch $\geq n-1$

Thm: every tree whose nodes include C_n, which sats $d_T(x,y) \geq d_G(x,y)$ must have

$$d_T(x,y) \geq \left(\frac{n}{3}-1\right) d_G(x,y)$$

for some x,y pair

Solution:

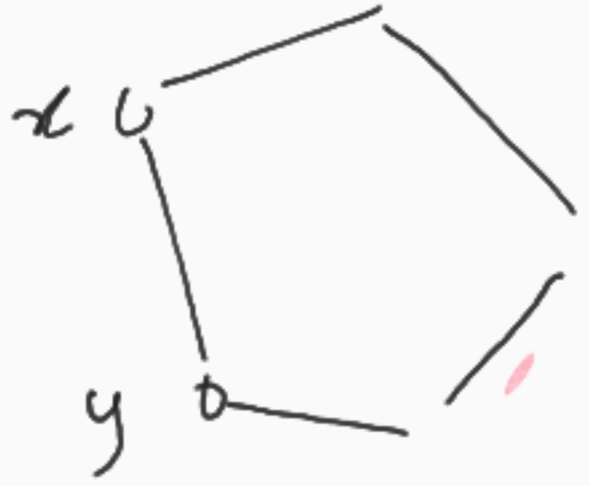
(Random) APSP trees:

$\mathcal{D}$ Distribution over trees T st

① $\forall T$ in supp$(\mathcal{D})$, $d_T(x,y) \geq d_G(x,y)$ $\forall x,y$

② $\forall (x,y) \in E(G)$, $\mathbb{E}[d_T(x,y)] \leq \alpha \, d_G(x,y)$.

Low Stretch Trees (Bartal '96, Fakcharoenphol, Rao, Talwar 03) FRT



$$\mathbb{E}\left[d_T(x,y) \right] = \frac{1}{n}(n-1) + \frac{n-1}{n} 1 = \frac{2(n-1)}{n}$$

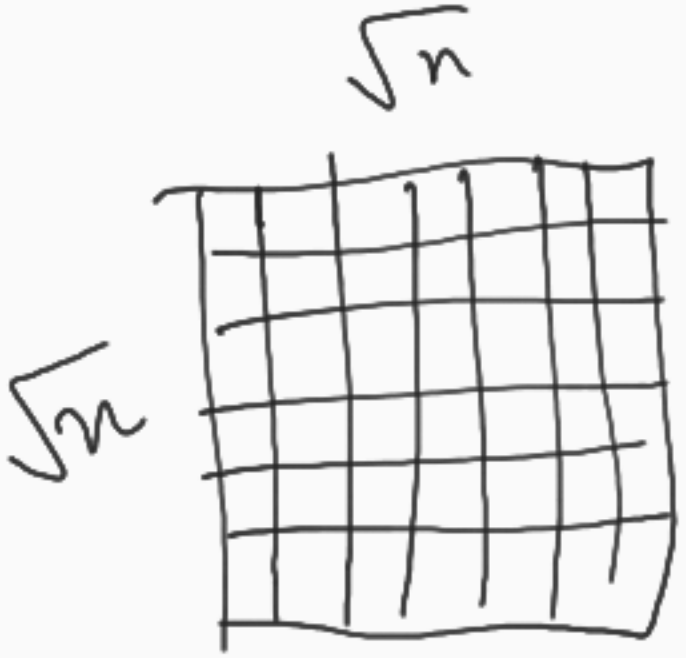
$$\leq 2$$

<u>Exercise</u>: if $E[d_T(u,v)] \leq \alpha \, d_G(u,v) \quad \forall$ edges $(u,v) \in E(G)$

$$\Rightarrow E[d_T(x,y)] \leq \alpha \cdot d_G(x,y) \quad \forall \ x,y \in V(G)$$

Thm: [FRT] ~~for ever~~ n vtx graph $\quad \alpha \leq O(\log n)$

best possible $\quad \exists$ graphs st $\alpha \geq \Omega(\log n)$



[Bartal] $\quad \alpha \leq O\left(\log n \cdot \underbrace{\log \frac{d_{max}}{d_{min}}}\right)$

"aspect ratio"

Δ = diam if $d_{min} = 1$

??

$\Longrightarrow$ distrib over subtrees of $\alpha_{ST} \leq \alpha_{NST} \cdot \log n$

Thm [Abraham Neiman]

$\alpha_{ST} \leq O(\log n \underbrace{(\log\log n)^2})$

- # of labeled sp. trees of K_n $= n^{n-2}$ [Cayley's Thm]
- doable using LinAlg. Kirchoff Matrix-Tree Thm
- Broder, Aldous, Wilson

(Low Diameter Decomp. LDD)

Graph G, param D

~~break~~ find a partition of V

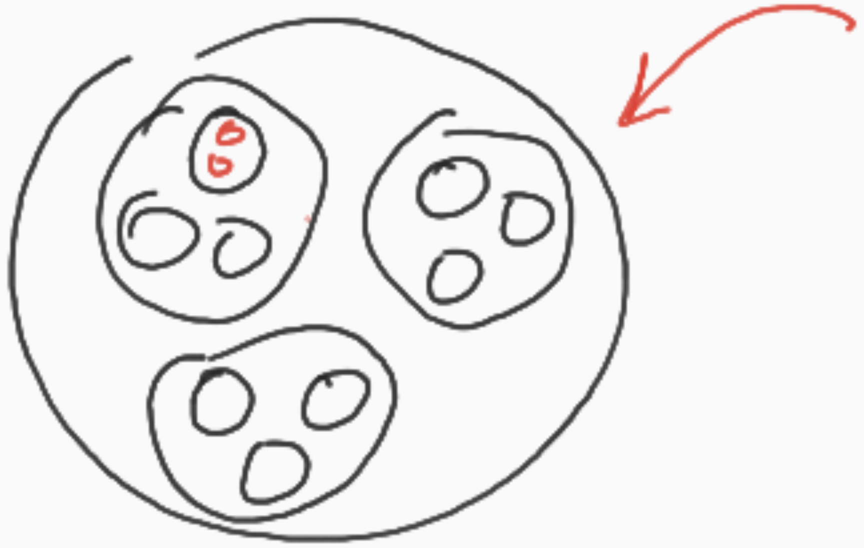
$$V_1, V_2 \dots V_s$$

s.t. (1) $\mathrm{diam}(G[V_i]) \le D$

(2) $\Pr((x,y) \text{ in diff pieces}) \le \frac{w_{xy}}{D} \cdot \beta$

$(x,y) \in E$

β ← $\log n$

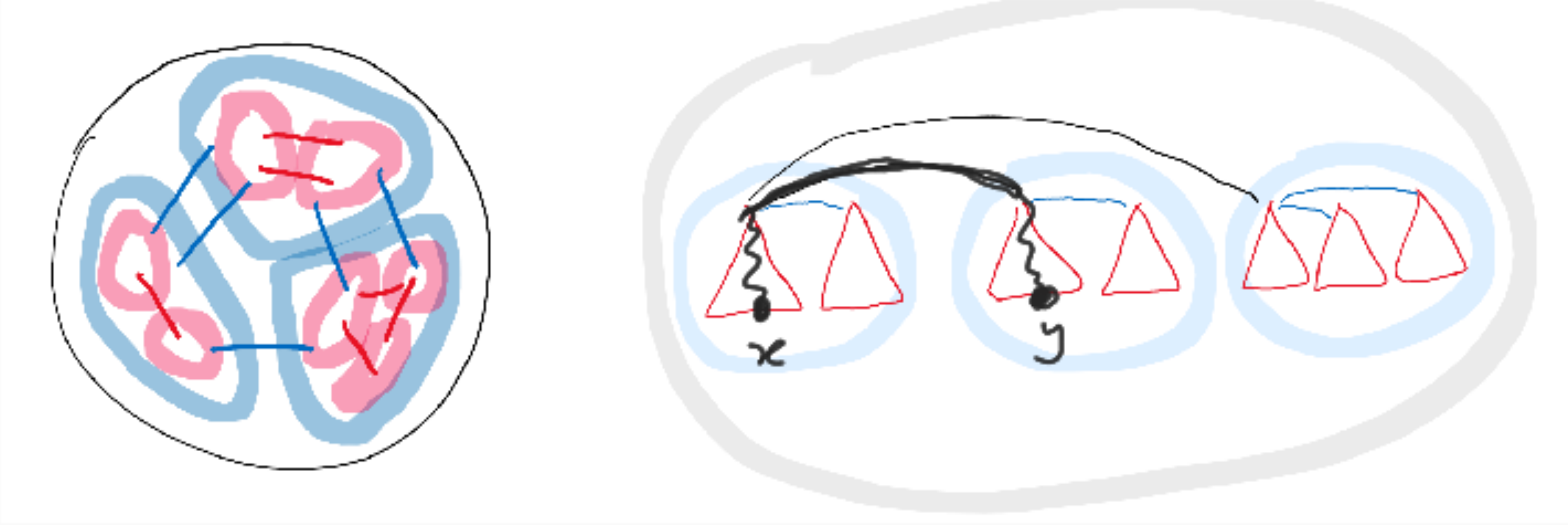


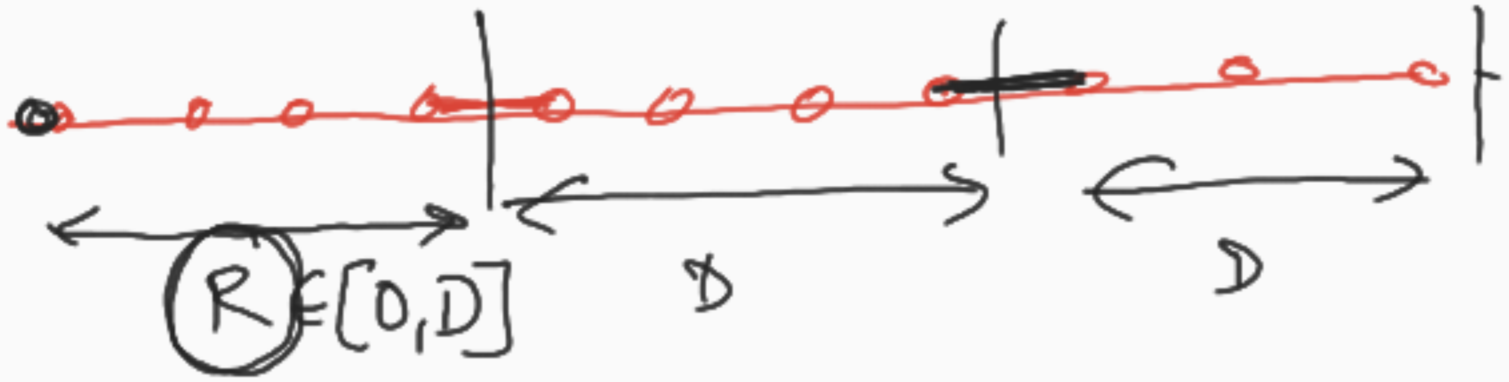
$O(\log n \lg \Delta)$

lose β at each "scale"

$\times$

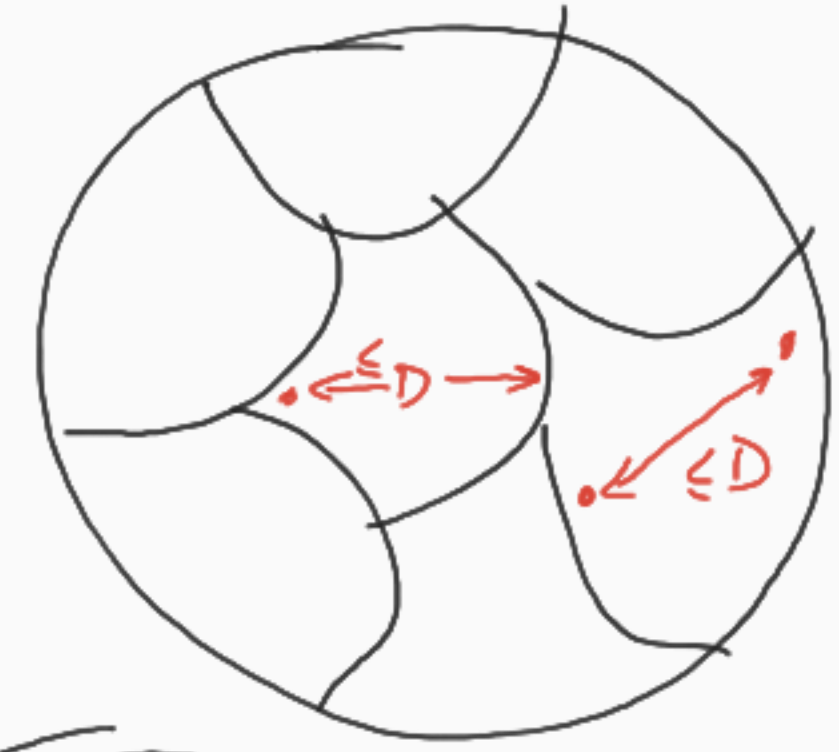
$(\lg \Delta)$ scales





Thm: G, n vtxs, $\exists$ LDD with $\beta = O(\log n)$

any D



Pf: Algo: while $G \neq \emptyset$, pick $v \in G$ arbit

$\left[R \sim \mathrm{Geom}\left(p = \frac{4 \log n}{D}\right) \right]$ $\min(-,1)$

$\boxed{\text{new cluster: } \{u : d_G(v,u) \le R\}}$

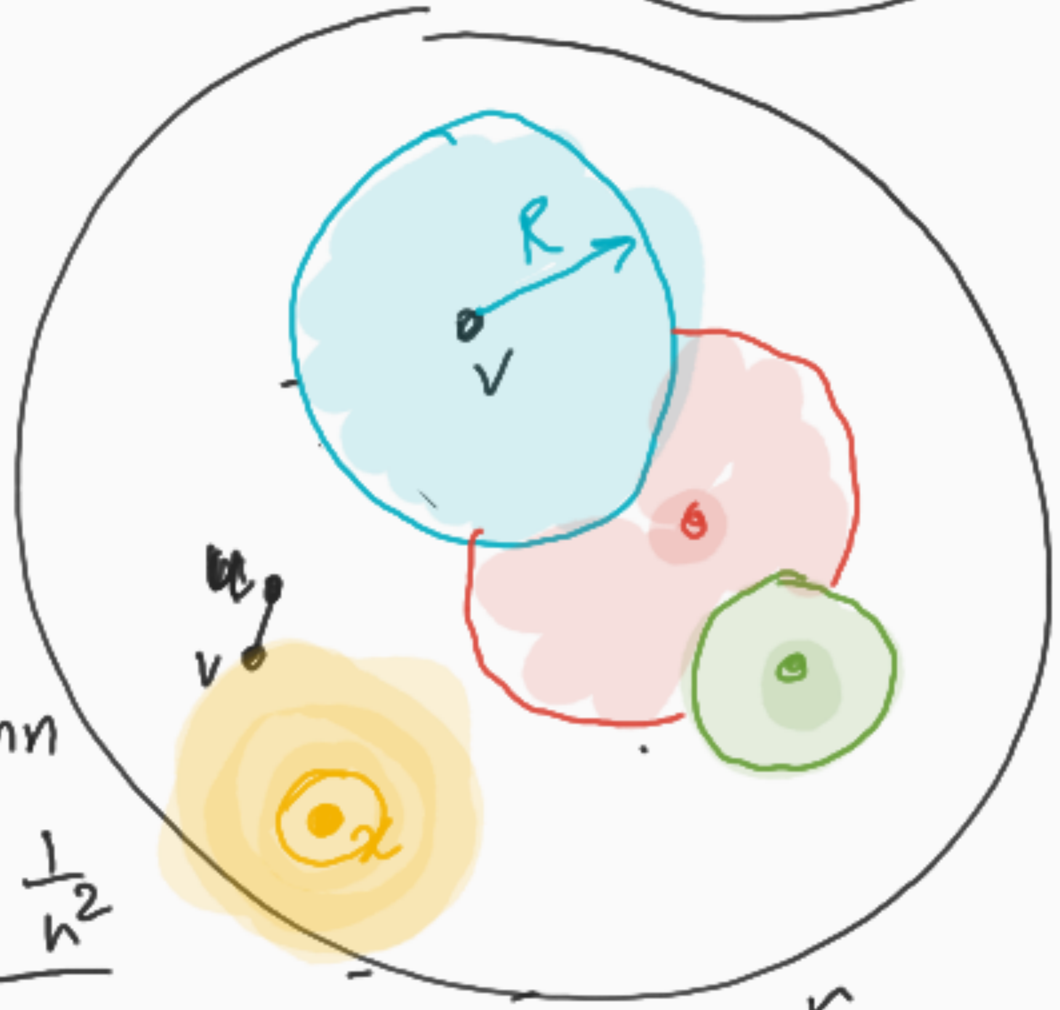
remove from G,

① Whp, diam(cluster) $\le D$

$$\Pr(\text{one } R > D/2) = \underbrace{(1-p)^{D/2} \le e^{-pD/2} = e^{-2\ln n}}_{= \frac{1}{n^2}}$$

$\Rightarrow$ by union bd, $\Pr(\text{all clusters} \le D) \ge 1 - \frac{n}{n^2}$

② $\Pr(u,v \text{ cut}) \le \frac{w_{uv}}{D} \cdot 4 \log n$



$1 + x \le e^x$