Lee 29: Recent Approximation Algorithms for TSP

- History (brief)
- The LP of interest
  - Better than 1.5 approx
    - Very high level ideas of proof
  - Open Qns.

- Christophides / Schederov
  (15) 79
- Karpin, Klein, Oveis Ghareh
  (15-8)
- \( \frac{4}{3} \)?
1.5 approx $G(V,E) \subseteq C_0$

1. MST in $G$: $c(T^*) \leq c(OPT)$

2. $O \subseteq$ odd degree vertices of $T^*$

3. $M^*$: Minimum perfect match on $O$

$$c(M^*) \leq \frac{1}{2} c(OPT)$$

$$c(T^* U M^*) \leq \frac{3}{2} OPT$$
**Thm:** $E[\text{cost}] \leq (1.5 - \varepsilon) \cdot \text{OPT}$

- **KKO (CSP)**
  
  **Algo:**
  
  1. Pick a random spanning tree of $G$
  2. $D \leftarrow$ odd degree
  3. $M^* \leftarrow \min \{ m \mid M \in D \}$
  4. output $(U, M^*)^T$

- Solve LP $x^*$ fractional
- Write $x^* \circ \sum_i k_i x_i^+ \in$ an even combo
- Sample from this
Graph: $K_n$

$|E| = n - 1$

$OPT = n$

$\text{MST} = n - 1 \geq \frac{3}{2}n$

$M' \leq \frac{n}{2}$

$\begin{align*}
\text{cost}(T) &= n - 1 \\
\text{cost}(M) &= (\text{Odd degree nodes in } T) \times \frac{1}{2}
\end{align*}$

$E[\text{cost of Alg}] = n + \frac{\text{#(odd degree nodes in } T)}{2}$

$E[\text{#odd}(T)] = (1 + e^{-2})n \approx 0.57n$

as $n \to \infty$
With LP for TSP:

\[ x^* \text{ LP soln.} \]

\[ x^* = \sum_i b_T x_T \]

max Entropy subject to:

\[ \text{sums up to } x^* \]

\[ |\mathcal{E}(T)| \sim \text{some value} \]

\[ \Rightarrow 57\% \text{ of vertices have odd degree} \]

Need to match them up.
\[
\begin{align*}
\text{min} & \quad \sum c_e x_e \\
\text{st.} & \quad \sum_{e \in \delta V} x_e = 2 \\
& \quad \sum_{e \in \partial S} x_e \geq 2 \\
& \quad x_e \geq 0
\end{align*}
\]

Fact: can find distrib of sparse trees s.t.
\[
\sum_{i} \phi_i x_i \leq x^*
\]

Draw a tree $T$ from this distrib.
Fact: $x^*$ is a fractional sol' to the following polytope:

$$\min \sum c_{ij} x_{ij}$$

$$\text{st } \sum_{e \in S} x_{e} \geq 1 \quad \forall \text{ cuts } (S, \bar{S})$$

$$\text{st } S \text{ contains one odd and one even degree vertex from } 0$$

$$\Rightarrow \text{ cost of } M \leq \text{ cost of TSP LP}/2$$

$$\Rightarrow 3/2 \text{ aprox again}$$
Shown: \( P_{\chi}(\text{vertex is even}) = 43\% \)

Want: \( P_{\chi}(\text{both endpoints of an edge are even}) \geq \text{constant} \)

Difficult parts of the proof 😞
Handwritten notes:

1. Shayan lecture notes
2. 2 talk videos

- Properties of max Entropy distributions
  - analogy of random spanning trees of $G$
- Properties of near-min cuts of Graph

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