Lec 29 : Recent Approximation Algorithms for TSP

- History (brief)
- The LP of interest
  - better than 1.5 approx
    - very high level idea of proof
  - open Qns.

- Christofides / Serdyukov
  (1.5) 79
- Karlin Klein Oveis Gharan
  (1.5 − ε)
  ↓
- 4/3 ?

1.5 approx $\quad G(V,E) \quad c_e$

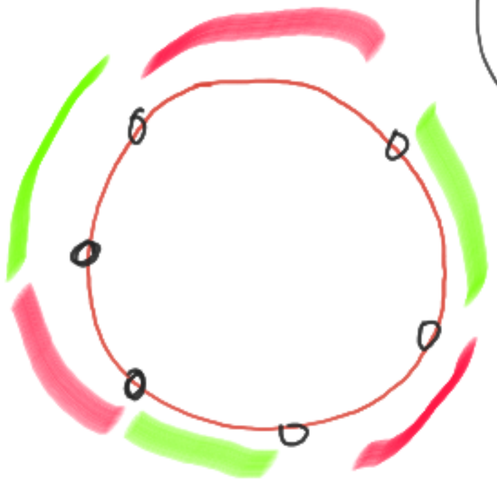
① MST on G $\qquad c(T^*) \le c(OPT)$

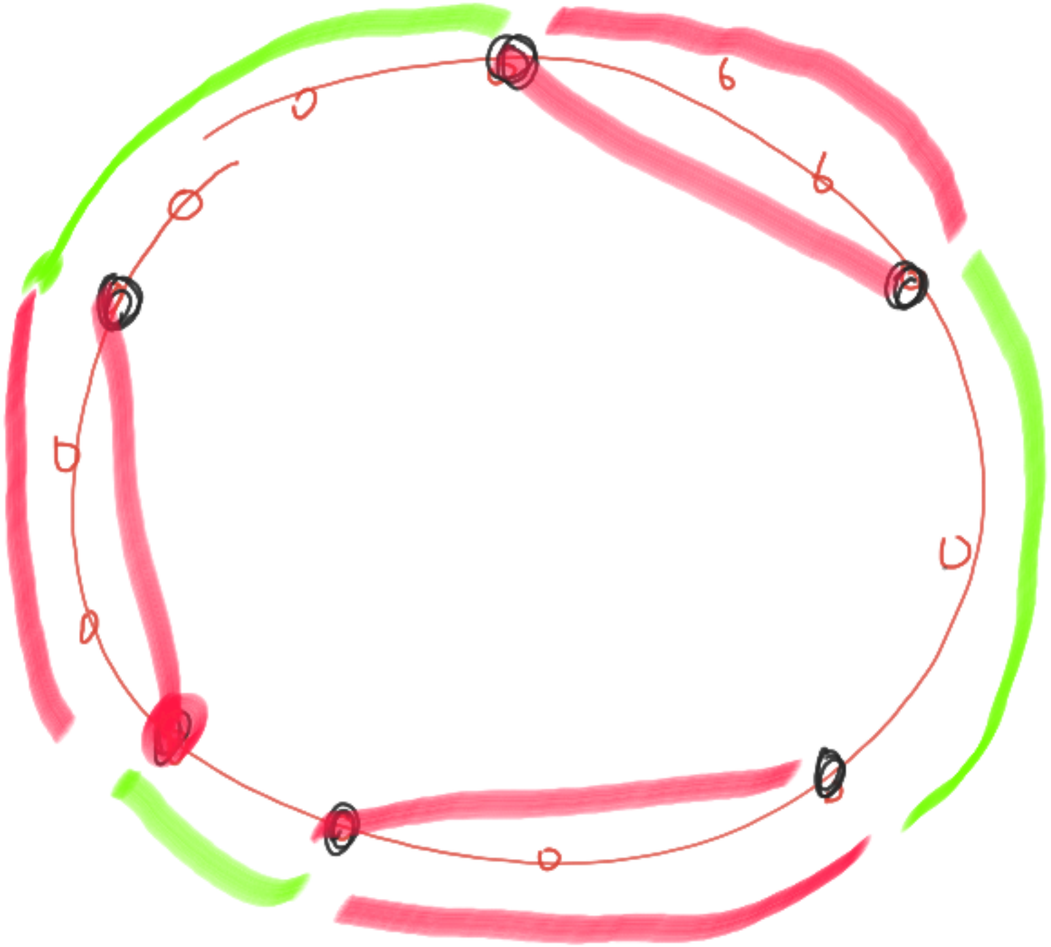
② $O \leftarrow$ odd degree vertices of T^*

③ M^* Min cost perf match on O

$$c(M^*) \le \frac{1}{2} c(OPT)$$

$$c(T^* \cup M^*) \le \frac{3}{2} OPT$$





Thm: $E[\text{cost}] \le (1.5-\varepsilon)\,OPT$

"most random" convex combo

Max Entropy combo

KKO (OSS)

Algo:

Pick a random spanning tree T of G

$O \leftarrow$ odd degree

$M^* \leftarrow$ min cost PM on O

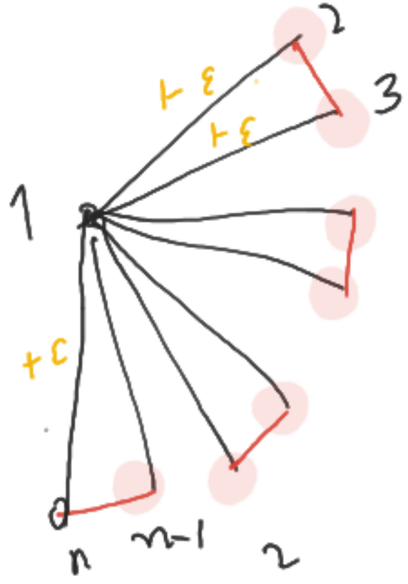
output $(T \cup M^*)$

- Solve LP x^* fractional tour
- write $x^* = \sum_T p_T \chi_T$ ← convex combo

sample from this

Graph: K_n OPT $= n$

costs $= 1$



$$\left.\begin{array}{l} MST = n-1 \\ M^* \leq n/2 \end{array}\right\} \frac{3}{2}n$$

Q: What if we pick a unif random spanning tree of K_n.

n^{n-2} distinct (labeled) sp. trees of

is actually what the KKO algo will do (pretty much)

$$cost(T) = n-1$$

$$cost(M) = (\text{Odd degree nodes in } T) \times 1/2$$

$$E[\text{cost of Alg}] \leq n + E\left[\frac{\#(\text{odd degree nodes in } T)}{2}\right] = 1.285n$$

Claim $E[\#\text{odd}(T)] = \left(\frac{1+e^{-2}}{2}\right)n \leq 0.57n$

as $n \to \infty$

Claim: $\mathbb{E}[\#\mathrm{odd}(T)] \le n\left(\frac{1+e^{-2}}{2}\right)$

random sp tree of K_n

Pf:

$$\sum_v \underbrace{\Pr[d_T(v) \text{ is odd}]}$$

$$= \Pr(Z_1 + Z_2 + \cdots + Z_d \text{ is odd})$$

$$= \Pr(\underbrace{Z_2 + \cdots + Z_d}_{} \text{ is even})$$

$$\le \Pr(\text{sum of } Ber(p) \text{ is even})$$

$\left(\frac{1-\cancel{2p}}{p}\right)$

[Hoeffding 56?]

1 Ber(1)

2 Ber(1/2) 1/2

3 Ber(1/3) $(2/3)^3 + 3(2/3)(1/3)^2 = 14/27$

$d_T(v)$

$= 1 + Z_2 + Z_3 + \cdots + Z_d$

$\sim \underbrace{p_2 + p_3 + \cdots p_d}_{= 1 - \frac{2}{n}}$

$\Pr(Poi(\lambda = 1) \text{ is even}) = \frac{1+e^{-2\lambda}}{2}$

to make this, all biases must $\in \{1, p, 0\}$

Thm: T be unif random sp tree of G ← any graph

$F \subseteq E(G)$

then $\underbrace{|F \cap E(T)|}_{\text{r.v.}}$ is a sum of $\le |F|$ Bernoullis independent

$= Z_1 + Z_2 + \cdots + Z_{|F|}$

$\Rightarrow d_T(v) = Z_1 + Z_2 + \cdots + Z_d$

$p_1 \ge p_2 \ge \quad \ge p_d$

b/c degree ≥ 1 wp 1 $\Rightarrow \underline{p_1} = 1$

$\mathbb{E}[d_T(v)] = \sum_1^d p_i$

$= \frac{2(n-1)}{n} = 2 - \frac{2}{n}$ (K_n)

Write LP for TSP.

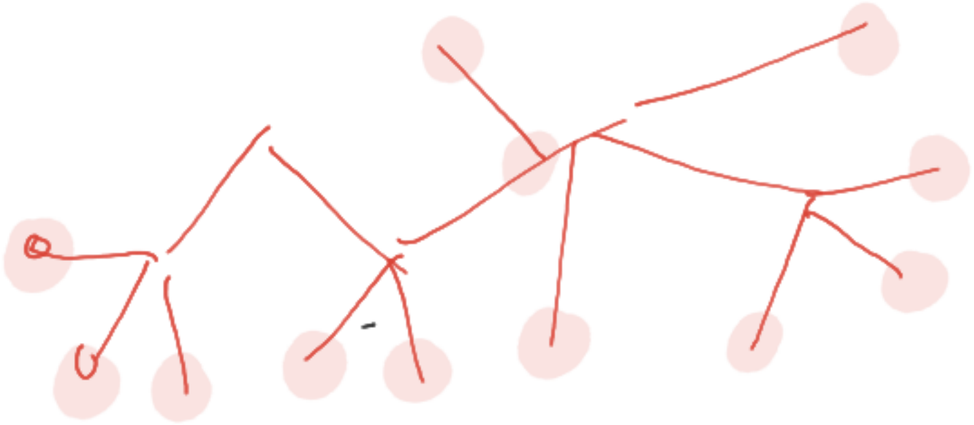
$\downarrow$

x^* LP sol^n.

$\downarrow$

$x^* = \sum p_T \chi_T$

max Entropy distrib that sums up to x^*



← 57% of vertices are odd

[all the properties we used above

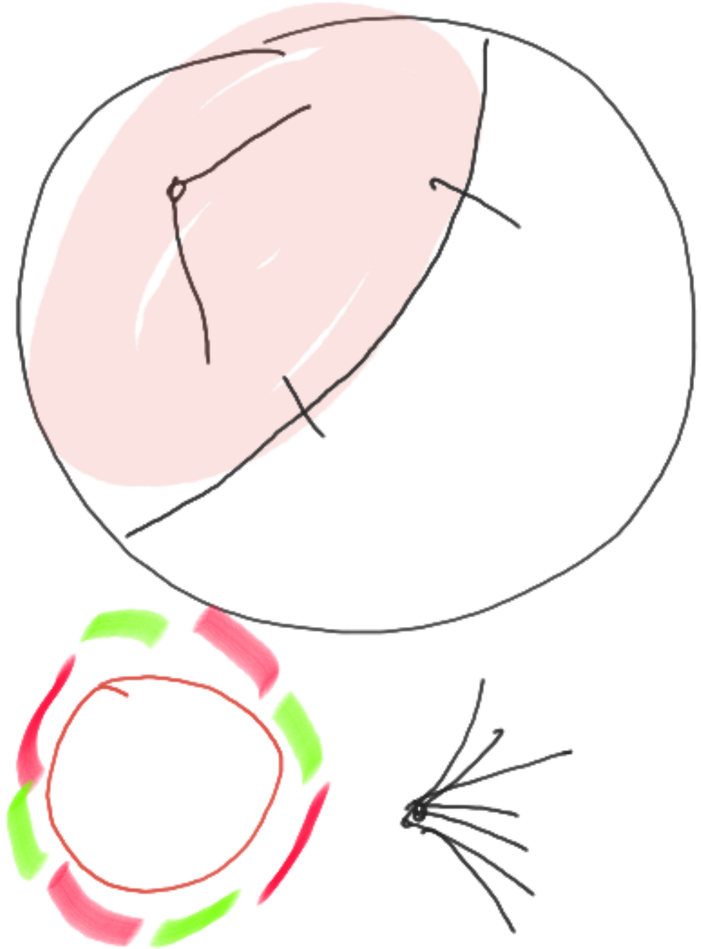
$|F \cap E(T)| \sim$ Sum of indep Bernoullis ✓

$\Rightarrow$ 57% of vertices have odd degree

Need to match them up.

TSP LP: var x_e

$$x^* \left\{ \begin{array}{ll} \min & \sum c_e x_e \\ \text{st.} & \sum_{e \in \partial v} x_e = 2 \\ & \sum_{e \in \partial S} x_e \geq 2 \\ & x_e \geq 0 \end{array} \right\}$$



Q: What is cost of the matching M^* on these 57%?

Ans: enough to show $\exists$ a cheap matching

$\Leftrightarrow$ $\exists$ a cheap fractional sol^n ~~matching~~ for perfect matching polytope

57% odd vertices

Fact: can find distrib of spanning trees st

$$\sum p_T \chi_T \leq x^*$$

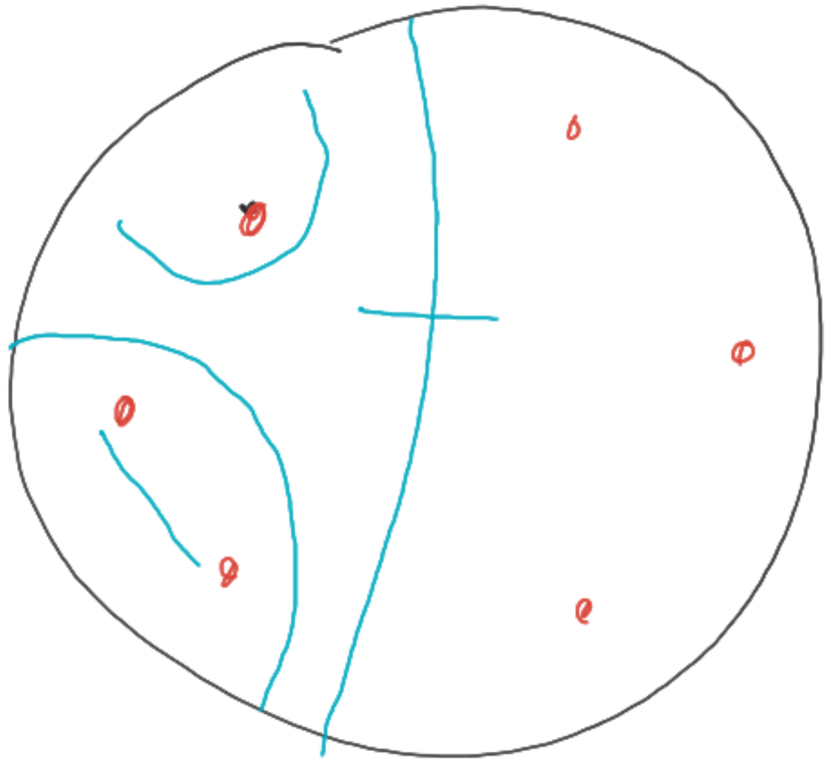
$\leftarrow$ MaxEntropy $\leftarrow$ draw a tree T from this distrib

Fact: $x^*/2$ is a fractional sol^n to the following polytope.

$$\min \sum_e c_e y_e$$

$$\text{st} \quad \sum_{e \in \partial S} y_e \geq 1 \quad \forall \text{ cuts } (S, \bar{S})$$

st S contains and odd # of vertices from O



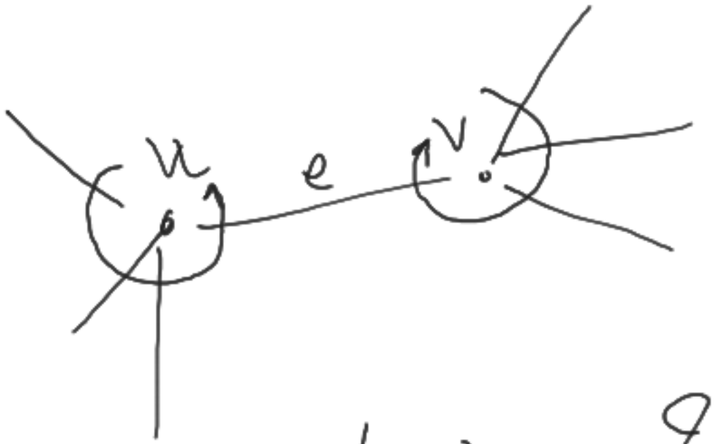
$\rightarrow$ LP for perfect matchg the O: odd degree vertices of T

$\Rightarrow$ cost of $M \leq$ cost of TSPLP$/2$ $\Rightarrow$ $3/2$ apx again

odd in T = 57%

<u>Showed</u>: $\Pr(\text{vertex is even}) = 43\%$

<u>Want</u>: $\Pr(\text{both endpoints of an edge are even}) \geq \text{Constant}$



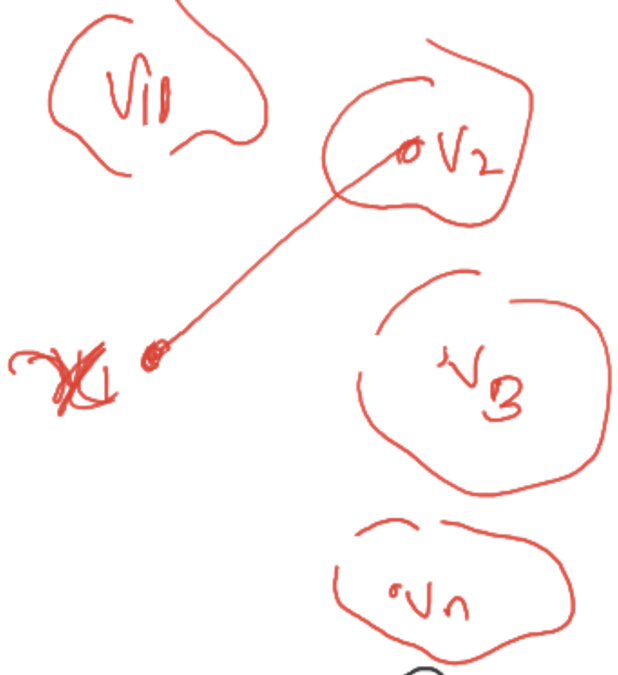
$\Pr(u \text{ is even } \& \text{ } v \text{ is even})$

Difficult parts of the proof ☹

Hand written notes

1. Shayan lecture notes

2. 2 talk videos ←



- Properties of max Entropy distributions
  ≈ analog of random spanning trees of G
- properties of near-min cuts of Graph

—— x ——