Lecture 27 Prophets & Secretaries (~~Experts, Bandits~~)

- Beyond the worst case #2
  - Bayesian Analysis and Random Order Models

- Problems and Solutions

Prophet Problem

auction

$X_1, X_2 \dots X_n$ distributions known

at each time t, see value of X_t
decide if to pick it / stop
or discard

* $X_1 \sim$ ~~Bent~~ Ber(0.3) $\begin{cases} 1 & \text{wp } 0.3 \\ 0 & \text{wp } 0.7 \end{cases}$

$X_2 \sim N(0,1) + 0.5$

$X_3 \sim \text{Geom}(0.7)$

$X_4 = 10$ wp 1

Online problem

$a_1 \quad a_2 \quad \dots \quad a_n$

Want Pick the largest #

· when you see #,
either discard it permanently
or pick it and stop

$$X_{max} = \max(X_1, X_2 \dots X_n) \longrightarrow \mathbb{E}[X_{max}] = OPT$$

E.g. $X_2 = \begin{cases} 1/\varepsilon & \text{wp } \varepsilon \\ 0 & \text{wp } 1-\varepsilon \end{cases}$ $\quad X_{max} = \begin{cases} 1/\varepsilon & \text{wp } \varepsilon \\ 1 & \text{wp } 1-\varepsilon \end{cases} \Rightarrow \mathbb{E}[X_{max}] = 1/\varepsilon \cdot \varepsilon + 1 \cdot (1-\varepsilon) = 2-\varepsilon$

$\boxed{X_1 = 1 \text{ wp } 1}$

Alg
- if pick, 1 for sure
- if pass (on X_1), get 1 in expectation

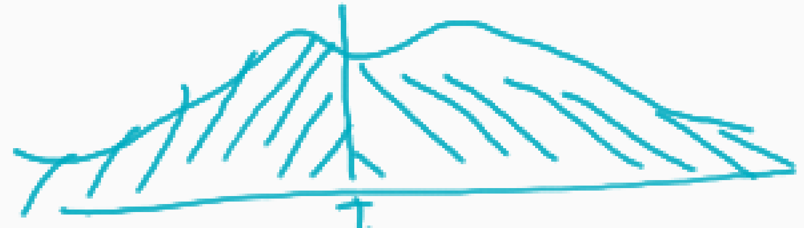
$\Rightarrow 1$ either way

[Thm]. No algo can get $> \mathbb{E}[X_{max}] \left(\frac{1}{2}+\varepsilon\right)$ for any $\varepsilon > 0$

[Thm] $\exists$ algo that gets $\geq \frac{\mathbb{E}[X_{max}]}{2}$

Algo 1: Define the median of X_{max} st $\Pr(X_{max} \geq \tau) = \Pr(X_{max} < \tau) = 1/2$

Pick 1st # bigger than τ ($\geq$). $\boxed{t}$ ← threshold



<u>Claim</u>: Algo (set median as threshold) gets value $\boxed{\frac{\mathbb{E} X_{max}}{2}}$ (Prophet Inequality)

$$(\_\_)^+ = \begin{cases} \_\_ & \geq 0 \\ 0 & < 0 \end{cases}$$

<u>Pf</u>: [E. Samuel-Cahn]

$$OPT = \mathbb{E}[X_{max}] \leq \tau + \mathbb{E}[(X_{max} - \tau)^+]$$
$$\leq \tau + \sum_i \mathbb{E}[(X_i - \tau)^+]$$

(Chawla Devanur Lykouris)

OR

$$Alg = \tau \underbrace{\Pr(X_{max} \geq \tau)} + \sum_i \mathbb{E}[(X_i - \tau)^+] \underbrace{\Pr\left(\bigwedge_{j<i} (X_j < \tau)\right)}$$

$\Pr$(none picked in 1st $i-1$ steps)

$\geq \Pr$(none picked in all steps)

$= \Pr(X_{max} < \tau)$

$$\geq \tau \cdot \underbrace{\Pr(X_{max} \geq \tau)}_{1/2} + \left(\sum_i \mathbb{E}[(X_i - \tau)^+]\right) \cdot \underbrace{\Pr(X_{max} < \tau)}_{1/2}$$

$$= \frac{1}{2}\left(\qquad \right) \geq \frac{1}{2} \cdot OPT.$$

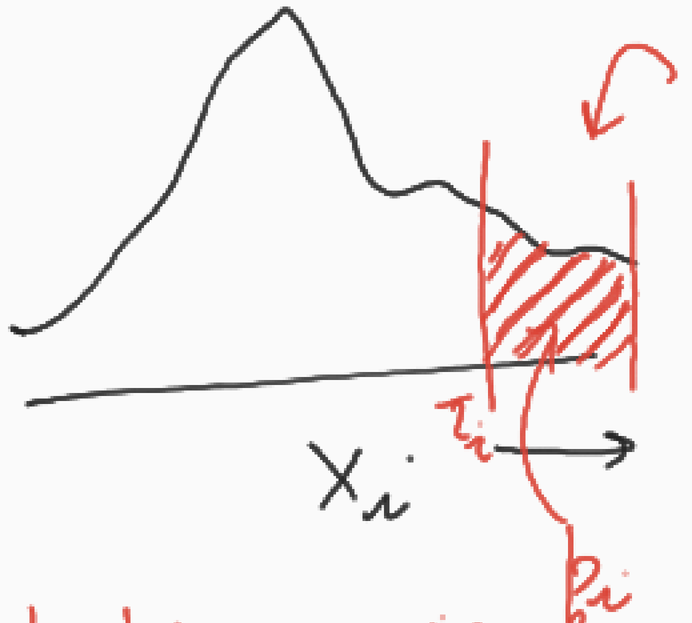
Multiple-item $\quad X_1\, X_2 \cdots X_n$

want to pick k items $\quad \left(\max \sum_{i \text{ picked}} X_i\right)$

<u>Algo #2</u> <u>for single item</u>

assume: no ties

Sps $\quad p_i := \Pr(X_i = X_{max})$ $\qquad \sum_i p_i = 1 \quad p_i \geq 0.$



$$OPT \leq \sum p_i \cdot v_i(p_i)$$

$\tau_i = 1 - p_i$ quantile for X_i

$\Pr(X_i \geq \tau_i) = p_i$

$\longrightarrow v_i(p_i) := E[X_i \mid X_i > \tau_i]$

<u>Algo</u>: Consider vars $X_1 \ldots X_n$

when at i, if not picked earlier,

if $X_i > \tau_i$, pick it w.p. $1/2$

$\Pr(X_i \geq \tau_i) = p_i$

Claim $\mathbb{E}[\text{val of Algo}] \geq \frac{OPT}{4}$

$\mathbb{E}[\text{val}] = \sum_i \Pr(\text{when reach } i\text{, not picked anyone yet}) \cdot \frac{1}{2} \cdot \underbrace{\Pr(X_i \geq \tau_i)} \cdot \underbrace{\mathbb{E}[X_i \mid X_i \geq \tau_i]}$

$\geq$

$\geq \sum_i \underbrace{\Pr(\text{no one picked at all})}_{\geq 1/2} \cdot \frac{1}{2} \cdot p_i \cdot v_i(p_i)$

$\geq \frac{1}{4} \cdot \underbrace{\sum_i p_i v_i(p_i)}$

$\Rightarrow$

$\geq \frac{1}{4} \cdot OPT$

☺

X_i

$p_i = \Pr(i^{\text{th}}\text{ item was max})$

τ_i

$v_i(p_i)$

$\Pr(\text{no one picked}) \geq 1/2$

$E[\#\text{people picked}] \leq \sum_i \underbrace{\tfrac{1}{2}} p_i = \frac{\sum_i p_i}{2} \leq 1/2 \Rightarrow \Pr(\geq 1 \text{ picked}) \leq 1/2$ (Markov)

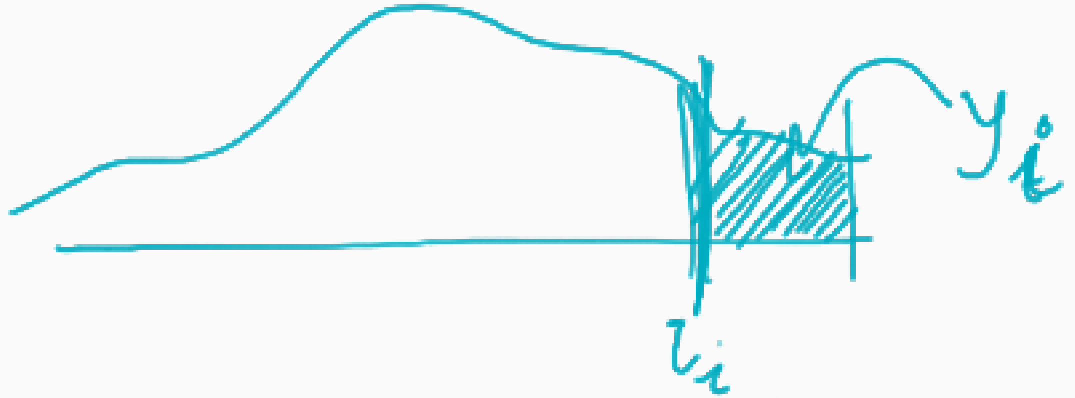
Convex Program

$$\max \sum_i y_i \, v_i(y_i)$$

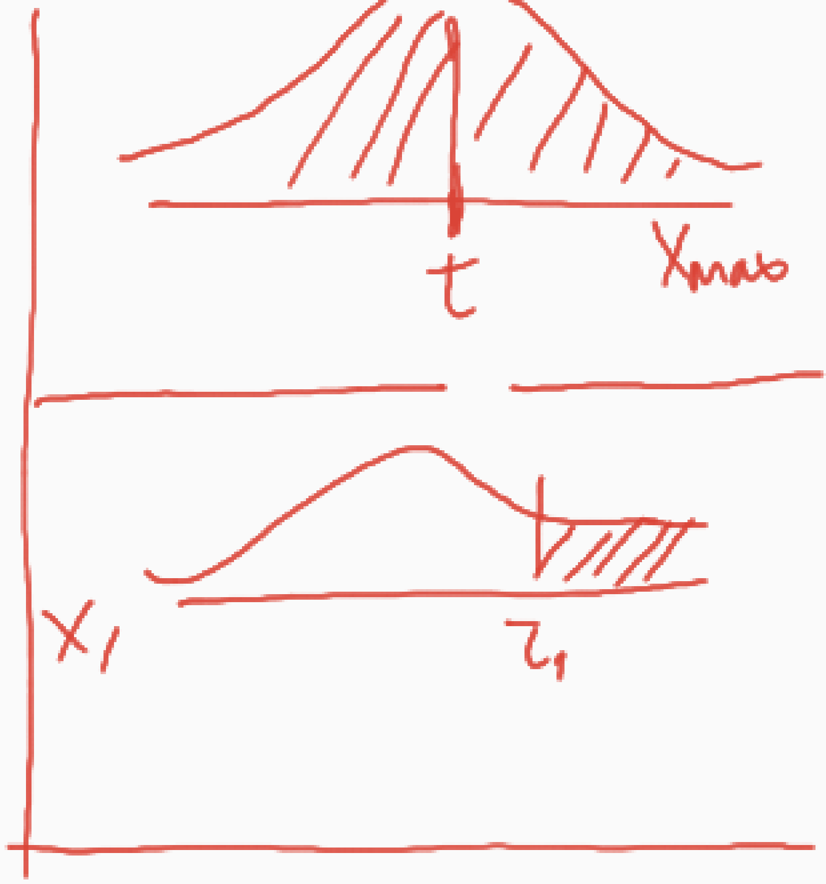
$$\sum_i y_i = 1$$

$$y_i \ge 0$$

(add more inequalities if needed)



$$v_i(y_i) = E[X_i \mid X_i \ge \tau_i]$$



Fact 1: $\forall_i$ soln to this, value $\ge$ OPT

Fact 2: Solve this CP

Fact 3: Proof shows.

$$\text{Alg} \ge \frac{1}{4} \text{ Convex Program value}$$

Improve: $\ge \frac{1}{2}$ OPT (simple)

Get k-item case (easily)

$\boxed{\text{Algo 3}}$ $X_1, X_2, \ldots X_n$

take 1 sample each, compute the max, use as threshold τ

Claim: 2-apx $\quad E[Alg] \geq \dfrac{E[X_{max}]}{2}$ (Exercise*)

sample efficient !!

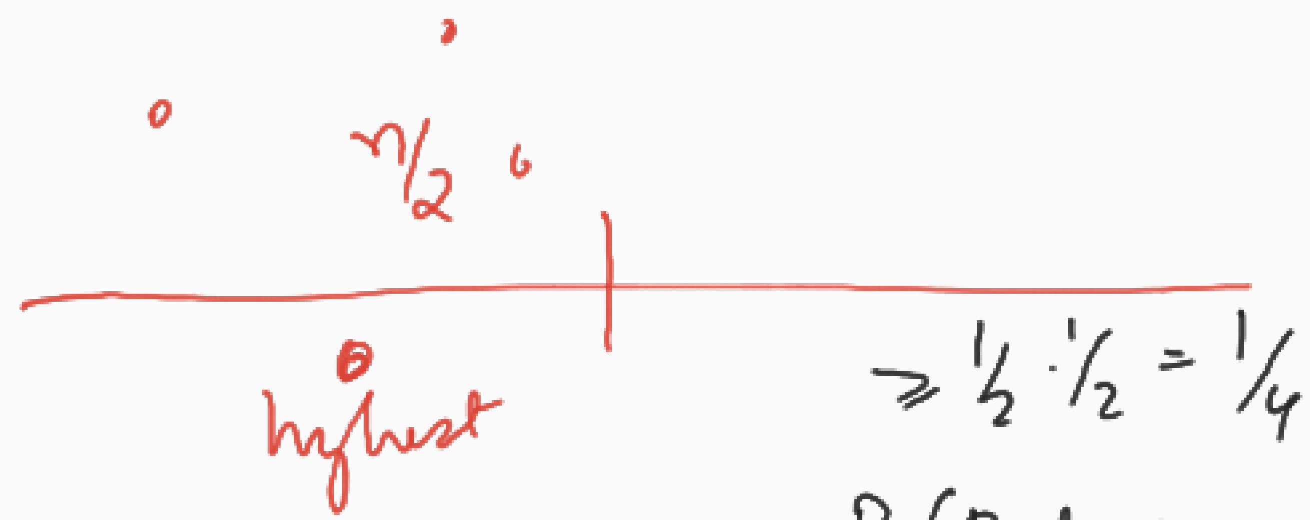
Secretary Problem:

Adv chooses $\boxed{n}$ numbers $v_1 \le v_2 \le \dots \le v_n$.

Presented to us in uniformly random order

$\underline{\max}$ Pr[picked v_{max}] ?

$\underline{\text{Thm}}$ [Dynkin] $\exists$ algo to get Pr( ) $\geq$ $\cancel{1/e}$ $1/4$ ~~and this is best possible~~



$\underline{\text{Algo}}$: wait until 50%

Pick anyone higher than best in 1st half

$\geq \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

Pr(pick v_{max}) $\geq$ Pr(2nd max in 1st half)

Pr(max in 2nd half | 2nd max in 1st half)

Pick a prefix max (but not too soon)

but only after seeing 50% of people

$$\Pr(\text{pick max}) = \sum_{i=s}^{n} \underbrace{\Pr(V_{max} @ i)}$$

$\Pr($max of 1st $i-1$ people lies in 1st s spots$)$

$$= \sum_{i=s}^{n} \frac{1}{n} \cdot \frac{s}{i-1}$$

$$= \frac{s}{n} \cdot \sum_{i=s}^{n} \frac{1}{i-1} \approx \frac{s}{n}(\log n - \log s) = \frac{1}{e} \quad \text{if } s = \frac{1}{e}n$$

$\blacksquare$

$$\max \quad 5x_1 + 3x_2$$

$$\begin{pmatrix} 0.5 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} x_1 + \begin{pmatrix} 0.2 \\ 0.3 \\ 1 \\ 0.2 \\ 0 \end{pmatrix} x_2 \leq \begin{pmatrix} K \\ K \\ K \\ K \\ K \end{pmatrix}$$

$$x_i \in \{0,1\}$$

$$\max \quad p^T x$$

$$\text{st } Ax \leq K\mathbb{1}$$

$$x \in \{0,1\}^n$$

if cols come in random order

$\Rightarrow$ get good algos

$$\max \quad v_1 x_1 + v_2 x_2 + \ldots v_n x_n$$

$$(1)x_1 + (1)x_2 + \ldots \ldots (1)x_n \leq K$$

$$x_i \in \{0,1\}$$