Lec 25: Online Algorithms

- Competitive Analysis
- Rent-or-Buy
- Paging
- other examples?

Sequential Decision Making

- $\boxed{\text{regret}}$ minimization
  - mistake bound model
- competitive analysis
  - competitive ratio

regret	comp ratio
∘ x_t before ℓ_t	σ_t before a_t
∘ best fixed action in hindsight	comp to $OPT(\sigma)$
∘ additive loss	mult loss

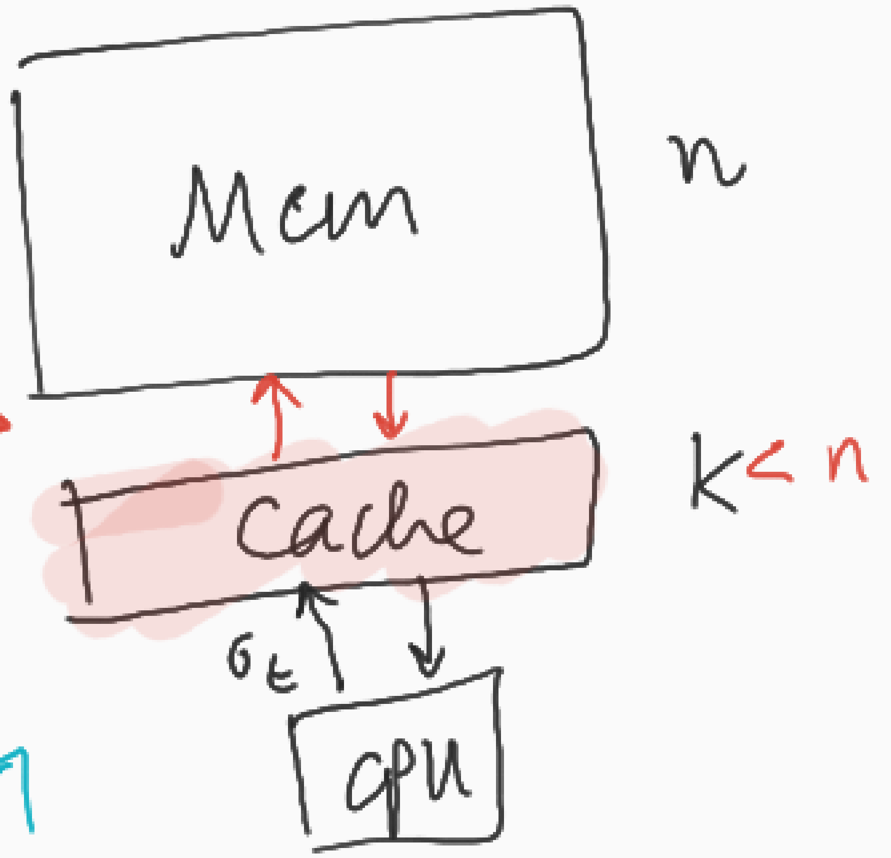
Model

Paging / Caching

at each time:

- get a request σ_t ← page requested
- a_t action taken by Algo. ← ~~none~~ page evicted / in cache
- cost of a_t for σ_t
  - ($\mathbb{1}$(if some page evicted))

cache miss



page replacement policy??

$\sigma = \sigma_1 \sigma_2 \ldots \sigma_T$

ALG(σ) = $\boxed{\text{total cost } (\sigma)}$ = #page evictions (σ)

OPT(σ) = min #page evictions on σ ← OPT = Furthest in Future

Competitive Ratio of Alg

$$\boxed{\rho = \max_{\sigma} \frac{ALG(\sigma)}{OPT(\sigma)}}$$

worst case

online algo for σ

optimal for σ

all request sequences

$\forall \sigma$

$ALG(\sigma) \le \rho \cdot OPT(\sigma)$

Sleator & Tarjan '84

Randomized Algo

$$\boxed{\rho = \max_{\sigma \in} \frac{\mathbb{E}[Alg(\sigma)]}{OPT(\sigma)}}$$

worst case over all inputs

algo is randomized

det quantity

- adv. fixes σ up front (oblivious adversary)
- (adaptive?? ← won't discuss these today)

Rent or Buy | Ski rental

rent \$1/day
buy \$B

length season unknown

$\overleftrightarrow{\underbrace{Y\;Y\;Y\;Y\;Y}_{\text{length}}}\;N\;N\;N\;N$ ——

Algos: rent for $i-1$ days then buy on day i $\quad A_1, A_2, \ldots, A_B, A_{B+1}, \ldots, A_\infty$

Claim: rent for $B-1$ days and then buy (A_B) has c ratio $2-\frac{1}{B}$
and this is optimal.

Algo A_B

Pf:

$$\frac{A_B(\ell)}{OPT(\ell)} = \frac{\text{if } \ell \le B-1 \text{ then } \ell \text{ else } \boxed{(B-1)} + \boxed{B}}{\min(\ell, B)}$$

$\xrightarrow{\ell \le B-1} \frac{\ell}{\ell} = 1$

$\xrightarrow{\ell \ge B} \frac{2B-1}{B} = 2 - \frac{1}{B}$

Best possible: for all poss $\boxed{\text{algo } A_i}$

show input I_ℓ st $\cancel{\text{cost}}$ $A_i(\ell) \ge (2 - \frac{1}{B})$ $OPT(I_\ell)$

Correct answer for det algo $= 2 - \frac{1}{B}$.

Q: do better for randomized algos?

Rand alg is a prob distrib over det algos $(B=4)$

I_j = season length j

A_i = buy on day i

		I_1	I_2	I_3	I_∞
p_1	A_1	$4/1$	$4/2$	$4/3$	$4/4$
p_2	A_2	$1/1$	$5/2$	$5/3$	$5/4$
p_3	A_3	$1/1$	$2/2$	$6/3$	$6/4$
p_4	A_4	$1/1$	$2/2$	$3/3$	$7/4$

Fact: I_∞ dominates $I_4, I_5 \ldots$

Fact2: if ... then no point playing $A_5 \ldots$

$p_1 + p_2 + p_3 + p_4 = 1 \qquad p_i \geq 0$

$4p_1 + p_2 + p_3 + p_4 \leq C$

$4p_1 + 5p_2 + 2p_3 + 2p_4 \leq 2C$

$4p_1 + 5p_2 + 6p_3 + 3p_4 \leq 3C$

$$C = \frac{1}{1-(1-1/4)^4} = 1/\left(1-\left(1-\frac{1}{B}\right)^B\right) \simeq \frac{e}{e-1}$$

Paging: n pages $\sigma_1, \sigma_2 \ldots\ldots \sigma_t \ldots\ldots$

k cache $k < n$ $\sigma_i \in [n]$

Deterministic: LRU, LFU, FIFO

Good/Bad: All of these are k-competitive (UB, LB)

Randomized Algos: $\boxed{\text{Randomized 1-Bit LRU}}$ $\boxed{\text{Random Marking}}$ ← optimal!

Good: $O(\log k)$ competitive

: No randomized algo can be better than $\Omega(\log k)$

Random Marking:

in each phase,
unmark all pages.

→ when get request σ_t

if $\sigma_t \notin$ cache

if $\exists$ no unmarked page, end phase.

else evict random unmarked page, bring in σ_t

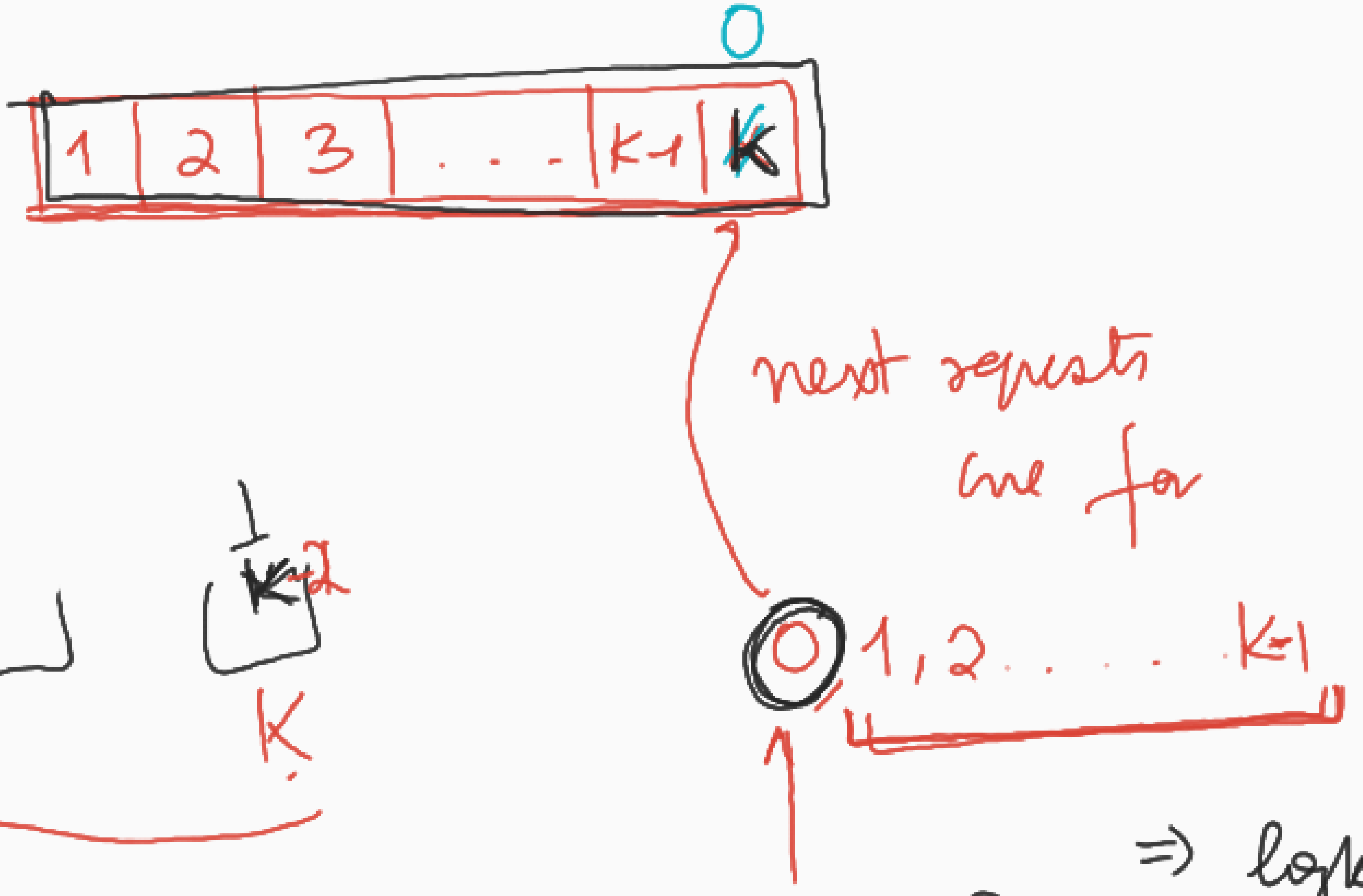
• mark σ_t

$K=4$

	*	*	
1	2	3	4

1 2 3 2 5 | 4 5 3 4 1 5 | 2 3

Airline Seat



How may people sit not in their own seat, in expectation?

$$\leq H_k = 1 + 1/2 + 1/3 + \cdots + 1/k$$

$\Rightarrow \log k$ in expectation

in each phase

Argue:

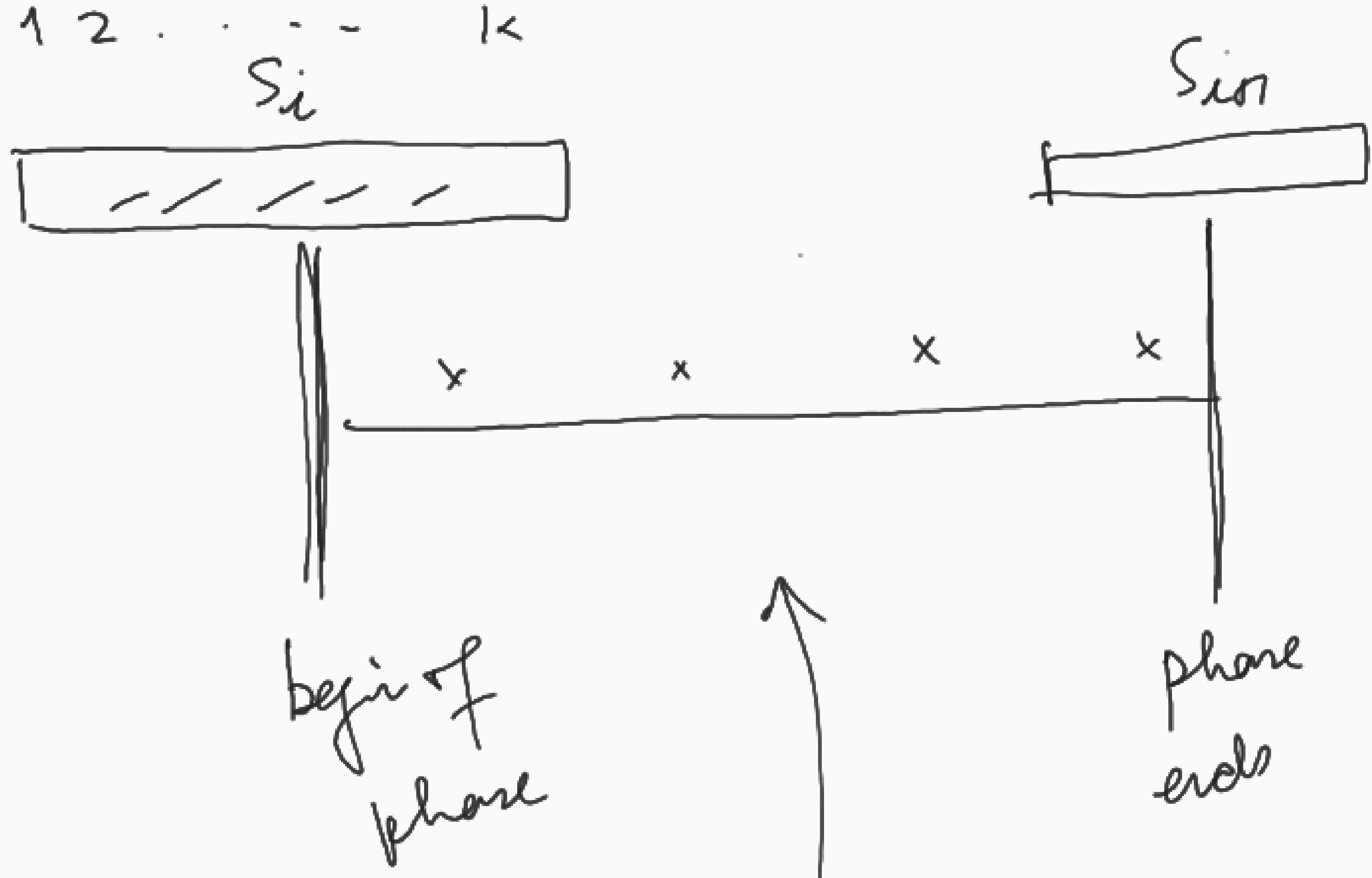
$E[\#\text{ evictions}] \le c \cdot \log k$

$=1$

OPT:

Random Marking

Sleator, Karlin, Fiat, Luby,



(k distinct pages requested here)

stale

c clean

0, 1 . . . , k-1

new

stale