Lec 25: Online Algorithms

- Competitive Analysis
- Rent-'n'-Buy
- Paying
- Other examples?

Sequential Revision Maley

- Regret minimization
  - Mistake bound model
  - Competitive analysis
  - Competitive ratio

Regret
- $r_t$ before $w_t$
- Best fixed action in hindsight
- Additive loss

Compete
- $T_t$ before $w_t$
- Comp to $OPT(t)$
- Mult loss
Model

→ at each time:
  get a request \( \sigma_t \)
  at a time taken by \( \text{CPU} \)

→ cost of \( \sigma_t \) for \( \sigma \)

\[ \sigma = \sigma_0, \sigma_1, \ldots, \sigma_t \]

\[ A(\sigma) \]

\[ \text{total cost } (\sigma) \]

\[ \# \text{pge evict } (\sigma) \]

\[ \text{OPT}(\sigma) = \min_{\sigma \in \sum} \# \text{pge evict} \text{ on } \sigma \]

Paying/Caching

→ page requested
→ page in cache
→ page evicted

Mem

\[ n \]

Cache

\[ k < n \]

CPU

Page replacement policy??
Competitive Ratio of Alg

\[ \rho = \max_\sigma \frac{\text{ALG}(\sigma)}{\text{OPT}(\sigma)} \]

worst case

all request sequences

online algo for \( \sigma \) optimal for \( \sigma \)

\[ \text{ALG}(\sigma) \leq \rho \cdot \text{OPT}(\sigma) \]

Randemized Alg

\[ \rho = \max_\sigma \frac{\mathbb{E}[\text{ALG}(\sigma)]}{\text{OPT}(\sigma)} \]

worst case on all inputs

alg is randomized

det quantity

\( \text{adv. fixes } \sigma \text{ up front (oblivious adversary)} \)

(adaptive?? \( \rightarrow \) worst, discuss these today)
Rent or Buy / Ski rental

rent $1/day  length season unknown
buy $B

Y Y Y Y Y N N N N

i=i

Always rent for days $A_1, A_2, \ldots, A_B, A_B+1, \ldots, A_k$
then buy on day $i$

Claim: rent for $B-1$ days and then buy $(A_B \leq)$ has crs ties $2-\frac{1}{B}$
and this is optimal
Pf:

\[
\frac{A_G(l)}{OPT(l)} = \begin{cases} 
\frac{l(l-1) \cdots (l-B+1)}{B!} & \text{if } l \leq B-1 \\
\min(l, B) & \text{otherwise}
\end{cases}
\]

Best possible for all pairs \( A_G, A_i \)

Show input \( I_e \) of \( \min(A_G(I_e)) \geq 2^{-B} \)

Correct answer for det \( A_{ge} = 2^{-\frac{1}{B}} \).

Q: do better for randomized algs?
Rand alg in a prob. dmhs on del edges

\[ I_g = \text{seam length} \]

\[ (B = 4) \]

<table>
<thead>
<tr>
<th>( A_i )</th>
<th>( I_1 )</th>
<th>( I_2 )</th>
<th>( I_3 )</th>
<th>( I_{\infty} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_1 )</td>
<td>1/4</td>
<td>2/4</td>
<td>3/4</td>
<td>4/4</td>
</tr>
<tr>
<td>( p_2 )</td>
<td>1/2</td>
<td>5/2</td>
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<td>1/2</td>
<td>6/3</td>
<td>6/4</td>
<td>4/4</td>
</tr>
<tr>
<td>( p_4 )</td>
<td>1/2</td>
<td>2/3</td>
<td>7/4</td>
<td></td>
</tr>
</tbody>
</table>

Fact 1: \( I_{\infty} \) always

\[ I_{p_i, I_5} \ldots \]

Fact 2: if \( g \) than no point pluses \( A_5 \) ...

\[ b_1 + b_2 + b_3 + b_4 = 1 \quad b_i \geq 0 \]

\[ 4b_1 + 2b_2 + 3b_3 + 2b_4 \leq C \]

\[ 4b_1 + 5b_2 + 2b_3 + 2b_4 \leq 2C \]

\[ 4b_1 + 5b_2 + 6b_3 + 3b_4 \leq 3C \]

\[ C = \frac{1}{1 - (1 - \frac{1}{2})^4} \quad \frac{c}{c + 1} \]

\[ \sqrt{c} \]
n pages
k cache k < n
\( \sigma_1, \sigma_2, \ldots, \sigma_t \ldots \)
\( \sigma_i \in [n] \)

Deterministic: LRU, LFU, FIFO

Good / Bad: All of these are k-competitive (UB)

Randomized Algorithms:

Randomized 1-Bit LRU

Good: \( O(\log k) \) competitive

Randomized Marley=

\[ \leq \text{optimal} \]

: No randomized algo can be better than \( O(\log k) \)
Random Markov:

in each phase,
unmark all pages.

\[ \rightarrow \text{When get request } \sigma_t \]

if \( \sigma_t \notin \text{cache} \)
\[ \left\{ \begin{array}{l}
\text{if no unmarked page, end phase.} \\
\text{else mark random unmarked page, buy in } \sigma_t \\
\end{array} \right. \]

mark \( \sigma_t \)

\[ k=4 \]
\[ 1 \ 2 \ 3 \ 4 \]

\[ 1 \ 2 \ 3 \ 5 \ 4 \ 3 \ 4 \ 1 \ 5 \ 2 \]
Airline Seat

How many people sit not in their own seat, in expectation?

\[ \leq H_K = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{K} \]
in each phase

Argue

$E[\# \text{ evictions}] \leq c \cdot \log k$

OPT:

Random Marley

Selector, Karp, Fiat, Luby,