Lecture 23: Approximation Algos.

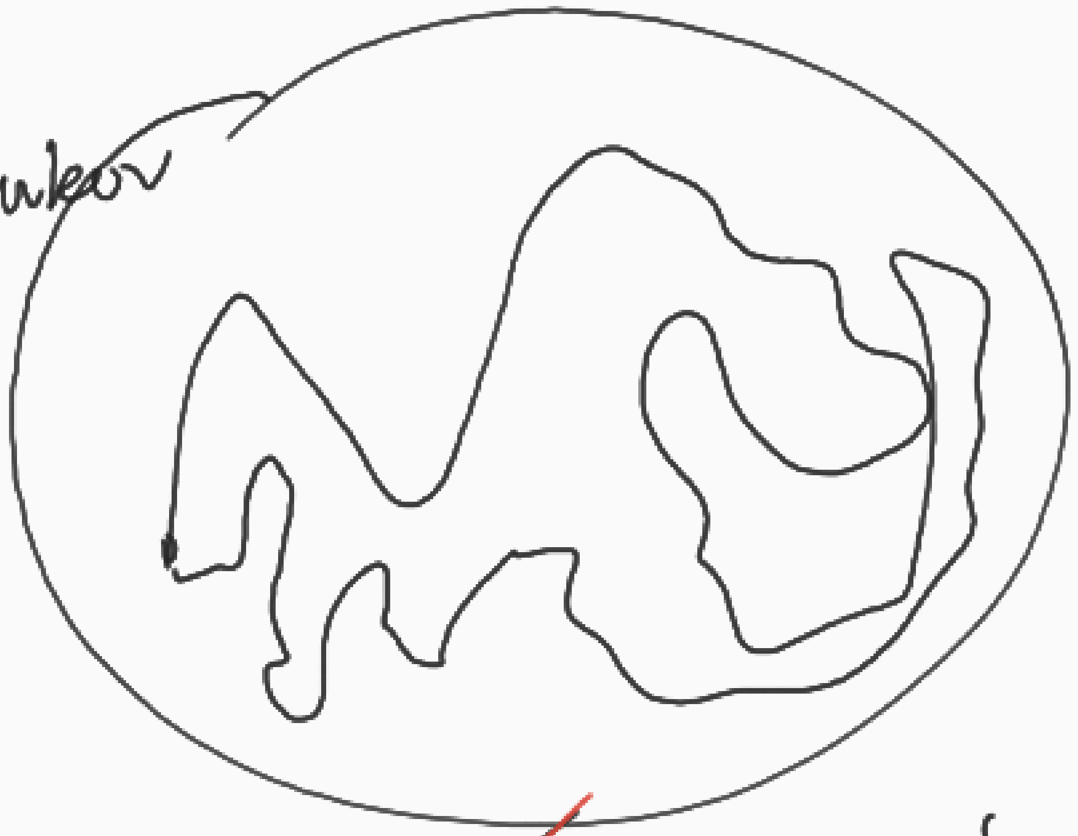
- NP-hard
- best we can in poly time

1976

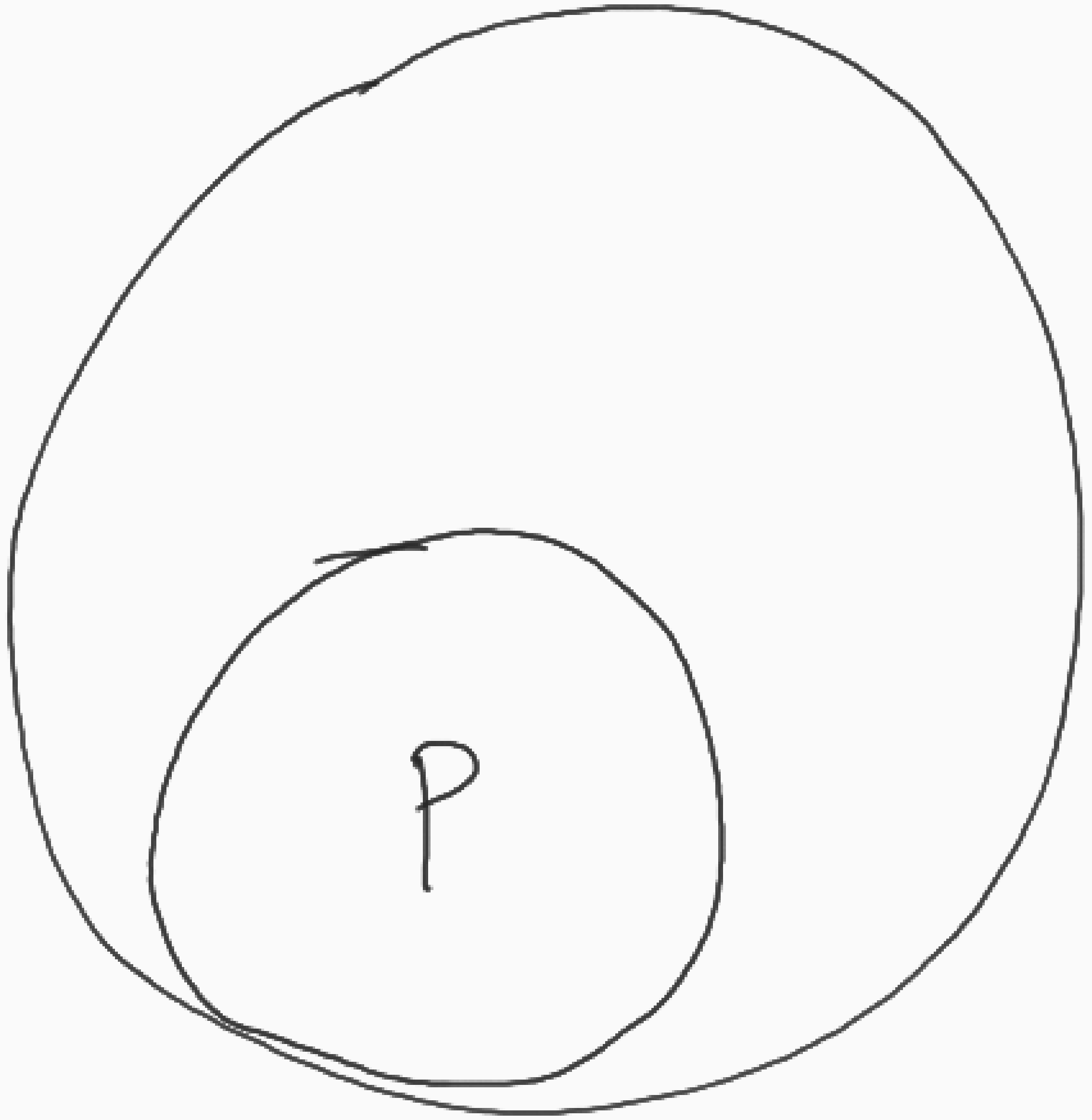
- Christofides | Serdyukov

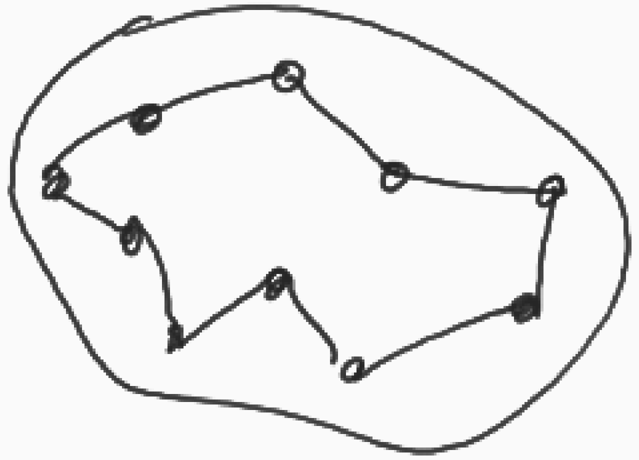
$$\left[\begin{array}{l} \text{cost}(\text{Alg}(I)) \\ \leq 1.5\ \text{cost}(\text{OPT}(I)) \\ \qquad \forall I \end{array}\right.$$

$1.5^{-\varepsilon}$ apx algo



✓ TSP (metric)





Problem TSP (minimization)

Instances I G, w_e

OPTimal $OPT(I)$

Algorithm $ALG(I)$

$$? \le \left\{ \frac{cost(Alg(I))}{cost(OPT(I))} \le \boxed{1.5} \right\}$$

1.5apx

$$cost(Alg(I)) \le 1.5\, \underset{cost}{(}OPT(I)) \quad \forall I.$$

Implicit: Alg run in poly time.

$$0.5 \le \frac{value(Alg(I))}{val(OPT(I))}$$

0.5 (best value) $\le$ myvalue

$\varepsilon > 0$

$(1+\varepsilon)$-apx

[FPTAS]

$(1+\varepsilon)$ apx

[PTAS]

$\rightarrow$ $O(1)$ apx

$\log^* n$

$\rightarrow$ $\boxed{O(\log n) \text{ apx}}$

$\log^3 n$

$\text{poly}(n)$ $\sqrt{n}$? $n^{1-\varepsilon}$

runtime

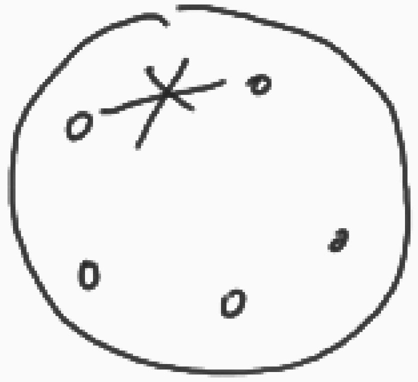
$\text{poly}(n, \frac{1}{\varepsilon})$

$n^{f(\varepsilon)} \nearrow$

$\varepsilon \to 0$

$f(\varepsilon) \to \infty$

poly



Bin Packing

Knapsack

TSP on Euclidean Plane

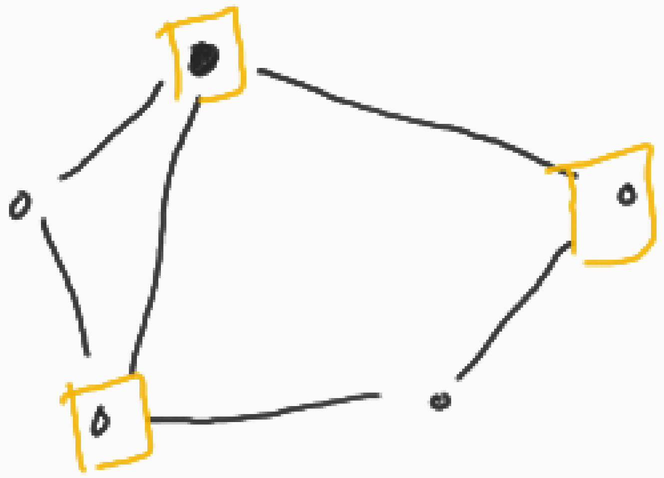
TSP on general metrics

1.5-eps, (1.01 apx $\Rightarrow$ P=NP)

$\boxed{\text{Set Cover}}$

$n^{1-\varepsilon}$

Problem Π (Set Cover)



- is Π in P or NP hard
  - yes (reduce from Vertex Cover on general graphs)

- Come up with algo

$$\text{Alg} \leq C \cdot \text{OPT}$$

- "Surrogate"

$$S(I) \leq \text{OPT}(I)$$

close

$$\text{Alg} \leq c \cdot S(I)$$

Universe U of elts $|U| = n$

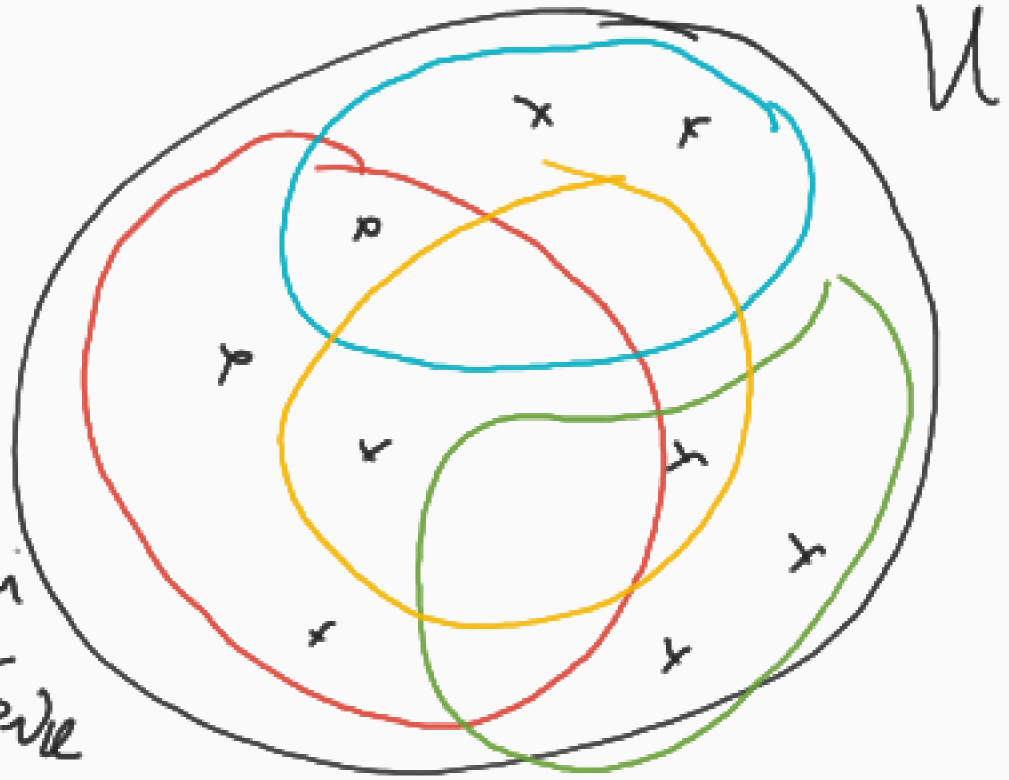
Collection of sets

$S_1 \, S_2 \, .. \, S_m$

s.t. $\bigcup_{i=1}^{m} S_i = U$

Find smallest collection of $S_{i_1}, S_{i_2} \, .. \, S_{i_k}$

s.t. their union $= U$



Algo for SC :

① Greedy.

repeatedly pick set that covers the most yet-uncovered elements

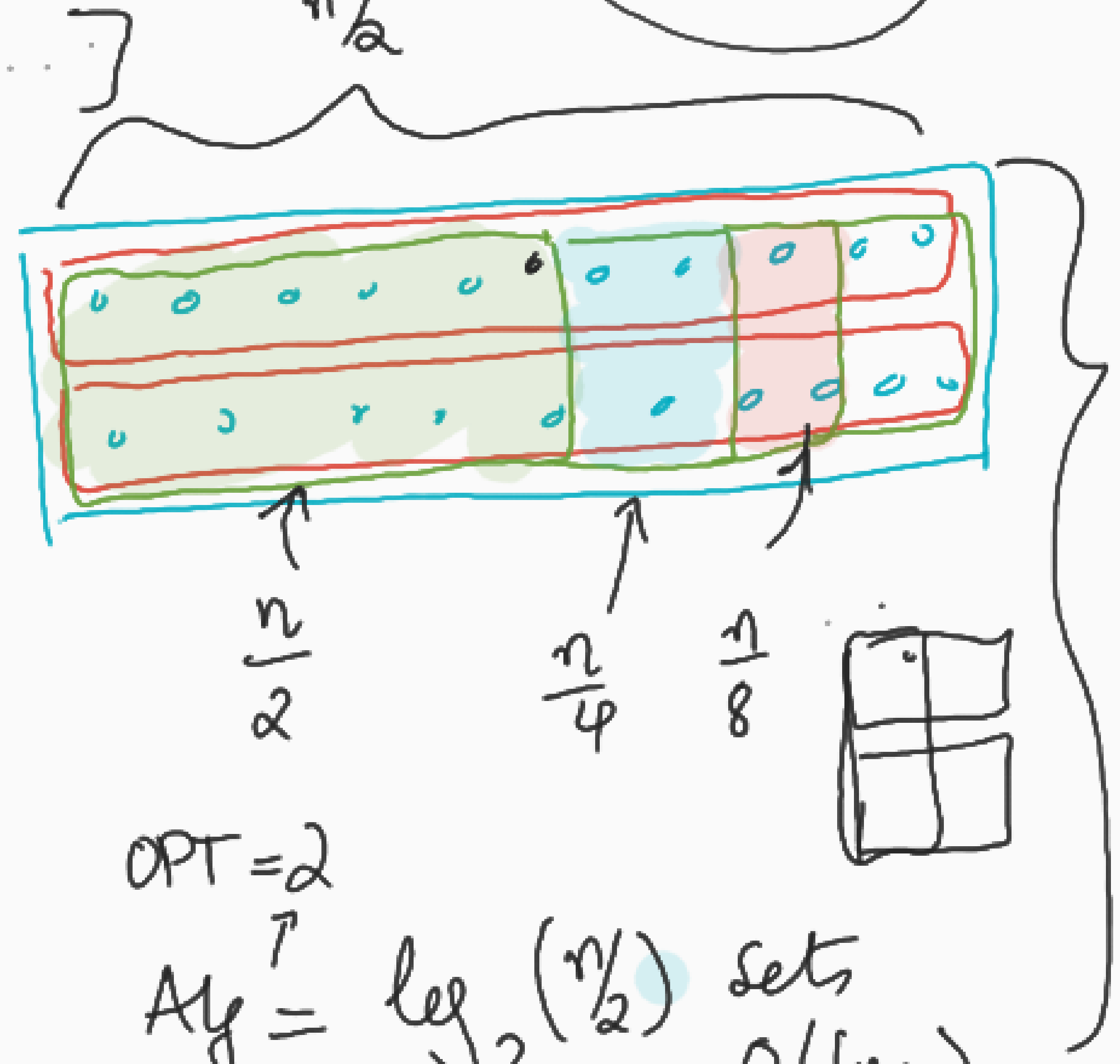
Thm. Greedy Algo is $(\ln n)$-apx. [Lovasz, Chvatal ...]

constant

$\exists \varepsilon > 0$ 0.99

Thm: if $\exists$ algo $(1-\varepsilon)(\ln n)$-apx $\Rightarrow$ RP = NP

(Feige, Dinur-Steurer) (random)



OPT = 2

Alg = $\log_2(n/2)$ sets $= \Theta(\log n)$

Pf: $\ln n$ apx

Sps OPT uses K sets

n_t = # of <u>uncovered</u> elts after alg picks t sets.

$n_0 = n$

$\exists$ set that covers $\geq \frac{n_t}{K}$ elts

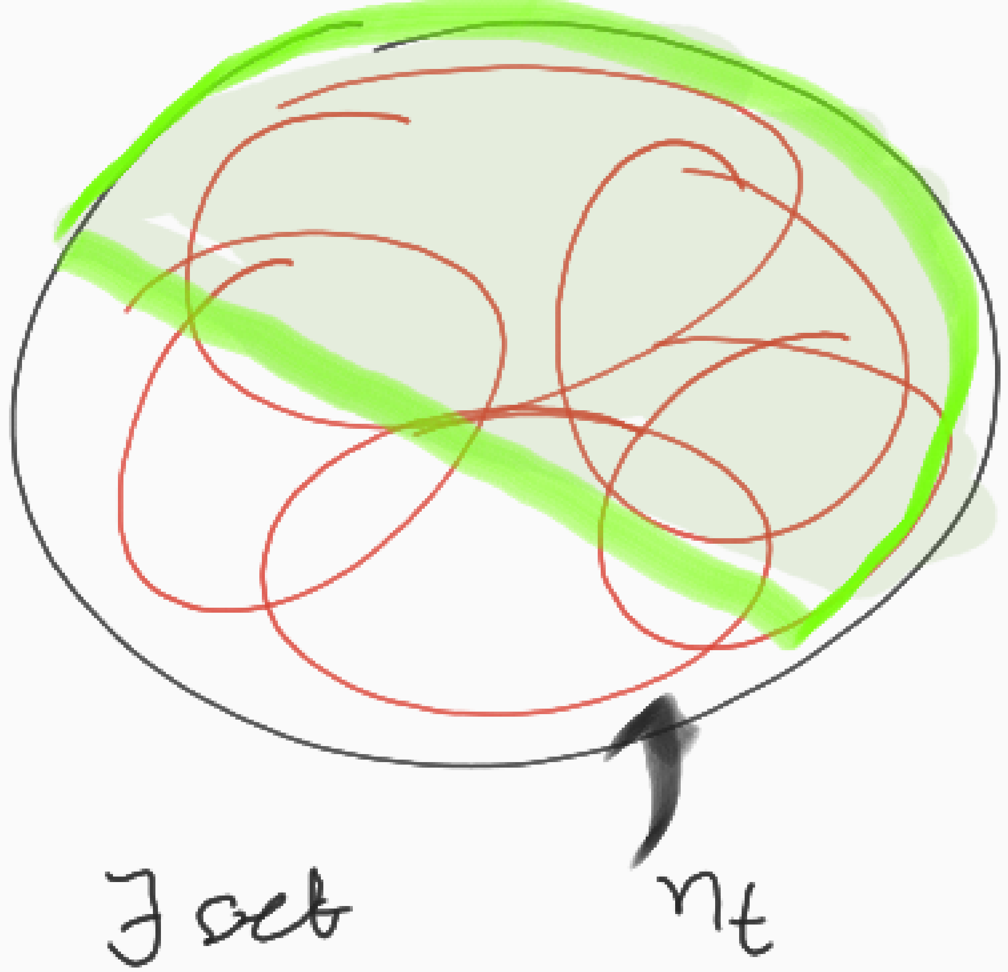
$\Rightarrow n_{t+1} \leq n_t\left(1-\frac{1}{K}\right)$

$\leq n_0\left(1-\frac{1}{K}\right)^t = n\left(1-\frac{1}{K}\right)^t < n \cdot e^{-t/K}$

$= n \cdot e^{-\ln n} = 1$

$1+x \leq e^x$
$<$

$\boxed{t = K \ln n}$



$\exists$ set covers $\frac{n_t}{K}$ elts.

Relax and Round

1. write an integer LP for problem.
2. relax to an LP.
3. "round" the fractional sol^n to LP into integer sol^{n}s

$$\text{ILP} = \boxed{\text{OPT}}$$
$$\text{ILP} \geq \boxed{\text{LP}}$$

$$\begin{aligned} \text{ALG} &\leq c \cdot \text{LP} \\ &\leq c \cdot \text{OPT} \end{aligned}$$

R-and-R for Set Cover

x_S $S \in$ collection

S_1	S_2	S_2
0.5	0.7	0.001

$$\begin{cases} \min \sum x_S \\ \text{st. } \boxed{\sum_{S: e \in S} x_S} \geq 1 \quad \forall e \in U \\ \cancel{x_S \in \{0,1\}} \quad x_S \in [0,1] \end{cases}$$

ILP
↓
LP soln $\boxed{\sum_S x^*_S \leq OPT}$

Round:

$\left(\text{Pick every set } S \text{ w.p. } x^*_S \text{ indp}\right) \times \ln n$ times

① $E[\text{cost of soln}] = LP \cdot \ln n$ ✓

② $\underbrace{\cancel{Pr[\text{soln is infeasible}]}}_{E[\#\text{ uncovered elts}]} = \sum_e \underbrace{Pr(e \text{ is not covered})}_{} = n \cdot \frac{1}{n}$

Pick 1 more set in expectation

$\left(Pr(e \text{ is not covered}) \right)^{\ln n}$

$= \prod_{S: e \in S} (1 - x^*_S) \leq e^{-\sum x^*_S}$

$\left(\leq e^{-1}\right)^{\ln n}$

$\leq 1/n$

<u>Algo</u>: Solve LP x_S^*

$\left(\forall \text{ set } S, \text{ pick it w.p. } x_S^*\right) \times \ln n$ tries

if $\exists$ uncovered elts, pick a set for each of them.

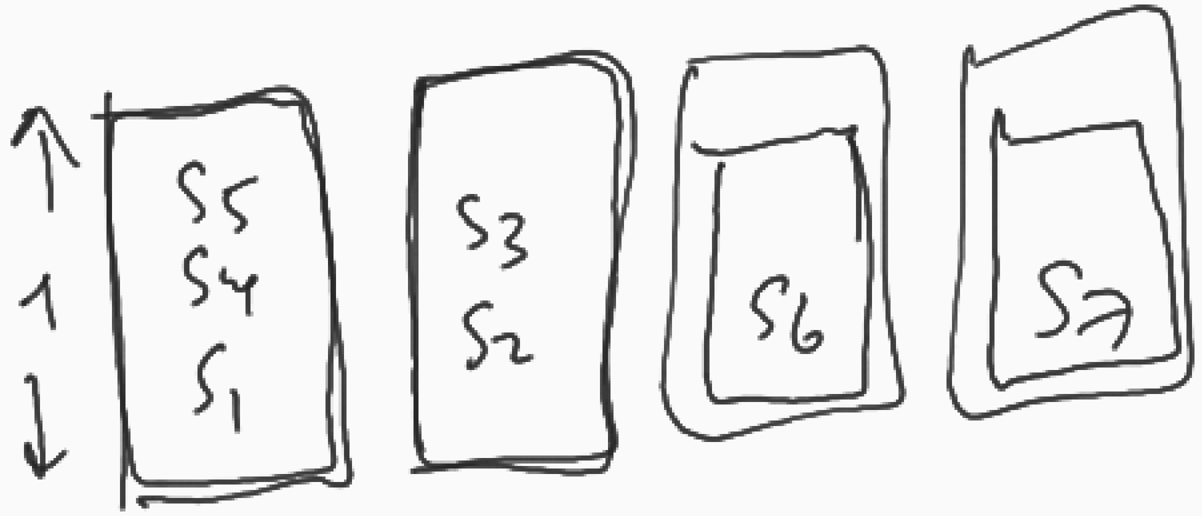
<u>Claim</u>: $\Rightarrow$ This algo has cost $\boxed{(\ln n)\,LP + 1}$ in expectation

always gives feas. sol[n].

Bin Packing

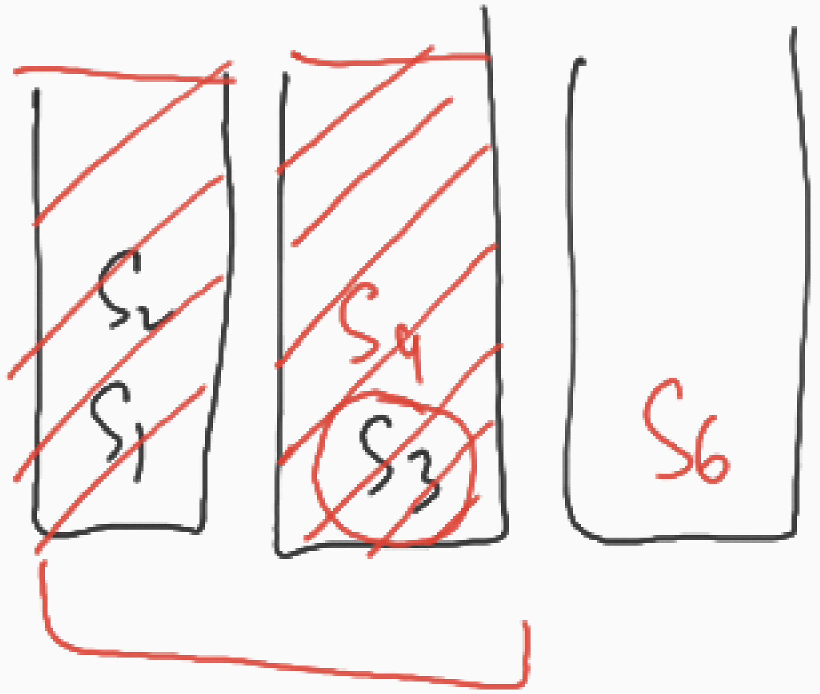
$s_1\ s_2 \ldots s_n$

$0 \le s_i \le 1$



min # bins

Surrogate $LB = \sum s_i = S$



$s_1\ s_2 \quad s_3\ s_4 \quad s_5$

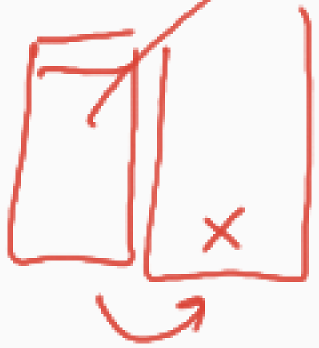
$0.3\ 0.5 \quad 0.3\ 0.2 \quad 0.6$

[First Fit]

[Next Fit] both 2 aprxs.

if k bins, $LB > \frac{k}{2}$

sps. all item $\le \varepsilon$ $\quad \ge 1-\varepsilon$



if <u>all items $\le \varepsilon$</u> $\Rightarrow$ all but ~~last~~ bin have $> 1-\varepsilon$ ~~items~~ volume in them.

$$\mathrm{vol} > (K-1)(1-\varepsilon)$$

$$K \le \frac{1}{1-\varepsilon} \cdot \mathrm{vol}(I) + 1 \quad \approx (1+\varepsilon)\,\mathrm{vol}(I) + 1$$

$$\le (1+\varepsilon)\ \mathrm{OPT} \ \boxed{+1}$$

Partition $a_1, a_2 \ldots a_n$ integers

$$\sum a_i = 2K$$

$$\sum_{i \in L} a_i = \sum_{i \in R} a_i$$

$$s_i = \frac{a_i}{K}$$

$\mathrm{OPT} = 2$

$(1+\varepsilon)\mathrm{OPT}$

$\Rightarrow$ distinguish b/w 2 bins & 3 bins is NP hard

$\Rightarrow$ get a $< \frac{3}{2}$ apx is NP hard.