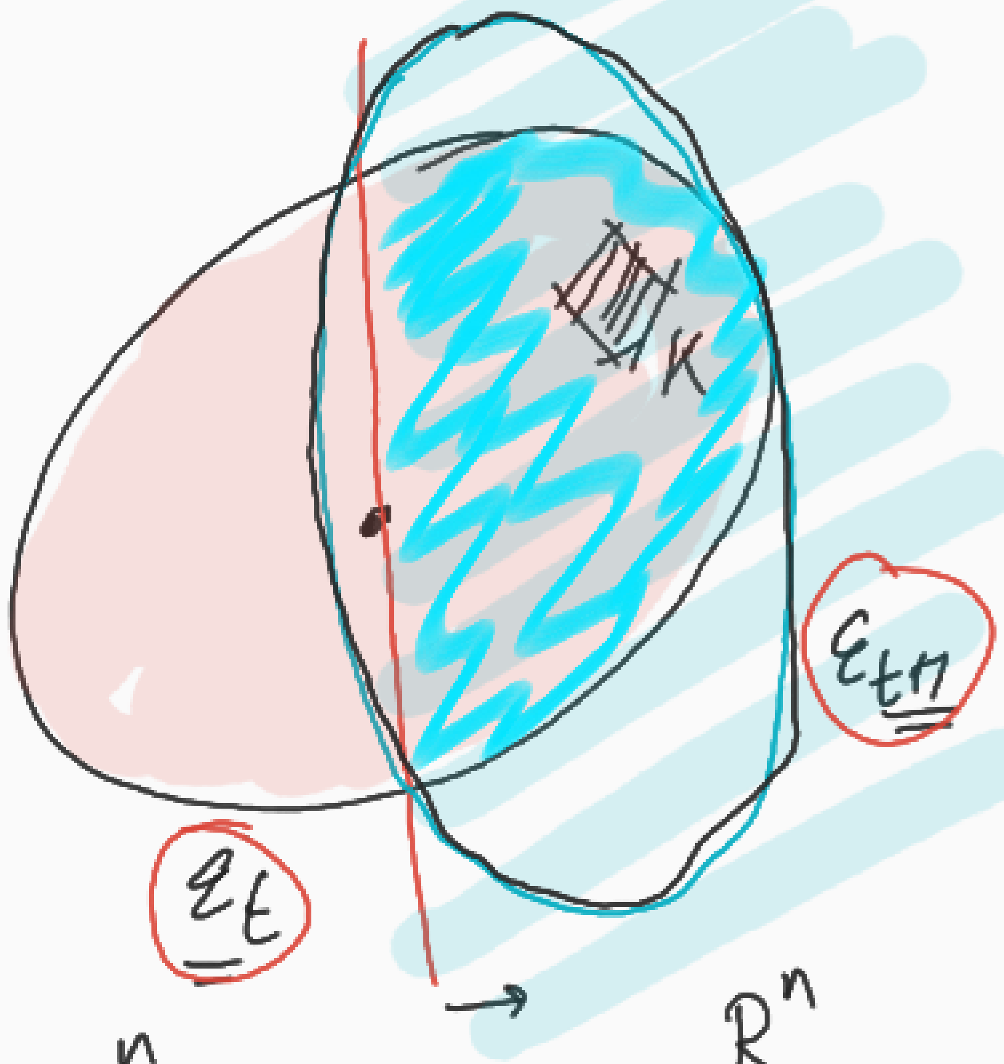
Lecture 21. Ellipsoid + Interior Point Algos

- Grötschel Lovász Schrijver
- Ellipsoid
  - (strong) separation oracle ←
  - main idea + basic theorems
- Interior Point
  - central path following algos.

$$\mathrm{vol}(\mathcal{E}_{t+1}) \le \mathrm{vol}(\mathcal{E}_t)\, e^{-\frac{1}{2(n+1)}}$$



<u>S.S.O.</u> K, $x \in \mathbb{R}^n$

① YES, $x \in K$

or ② No, H halfspace $K \subseteq H$, $x \notin H$

$$\underset{\sim}{r^n} \quad\quad R^n$$

$$B(c,r) \subseteq K \subseteq B(0,R)$$

$$e^{-\frac{1}{2(n+1)}} \simeq \left(1 - \frac{1}{2(n+1)}\right) \le \frac{n}{n+1}$$

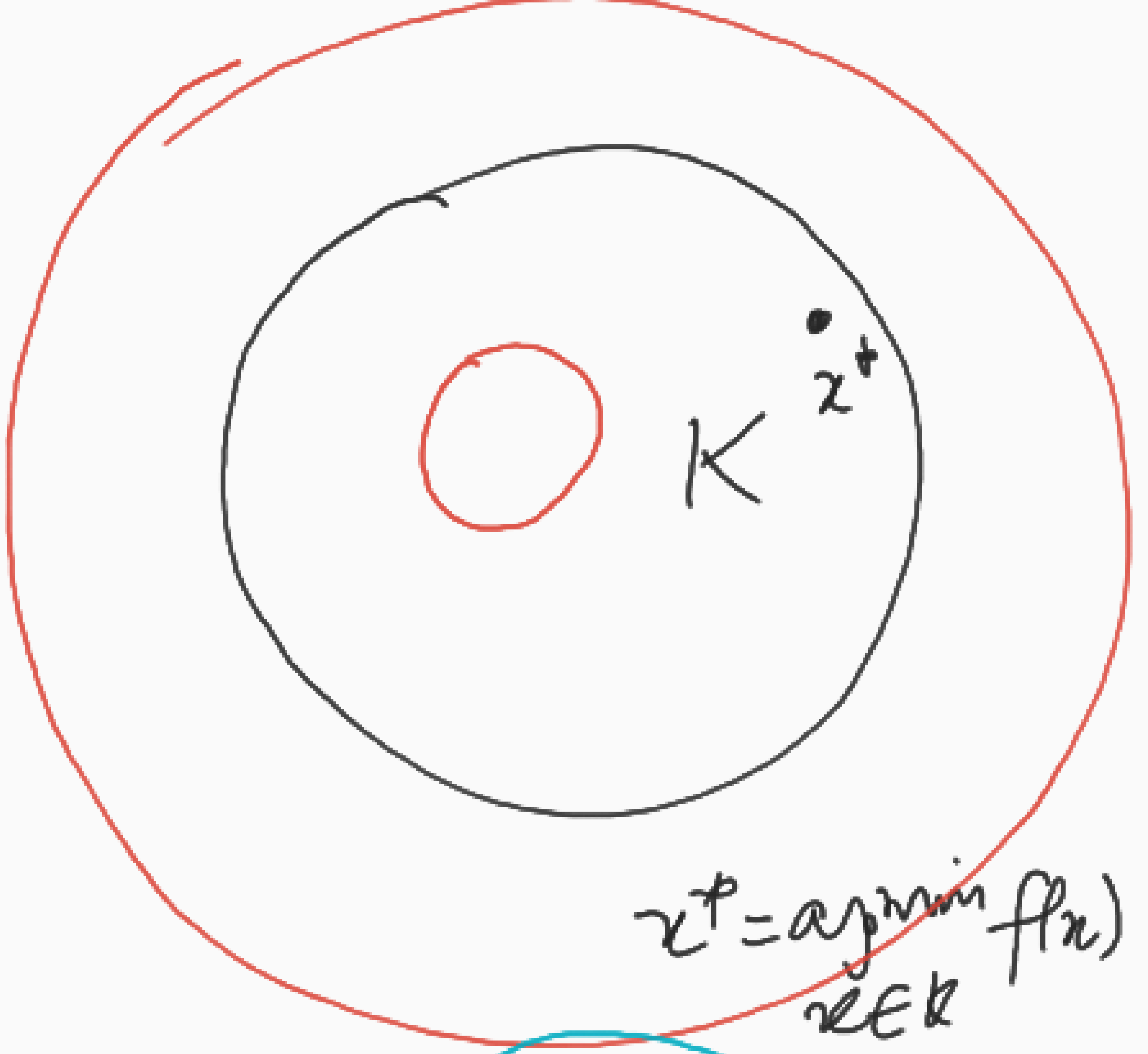
Thm 0: Given a convex fn $f : K \longrightarrow [-B, B]$

over convex set K

can find $f(\hat{x}) \leq f(x^*) + \epsilon$

after $T = O\left(n^2 \log \frac{BR}{r}\right)$

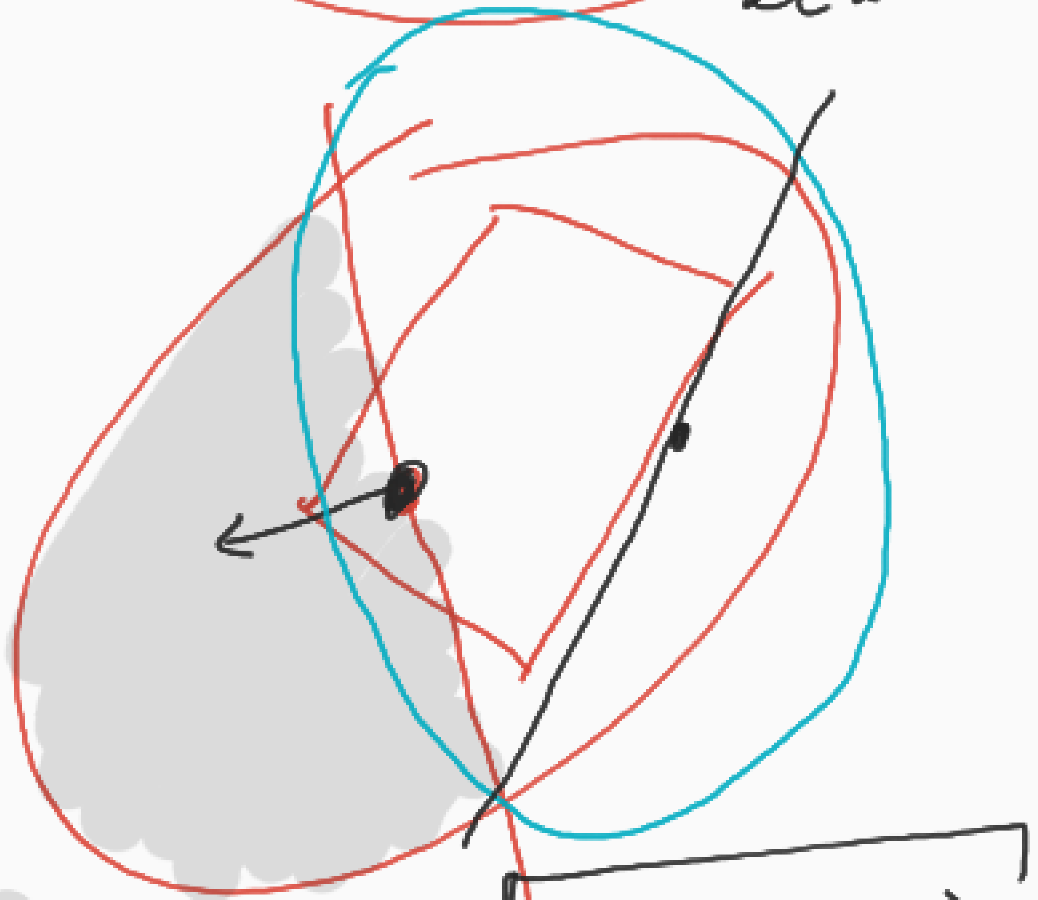
where $B(c, r) \subseteq K \subseteq B(0, R)$



<u>Here</u> given f via value oracle & gradient oracle

K via strong separation oracle.

given r, R.



$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle$

Thm 1: Given LP explicitly $\boxed{\begin{aligned} \min\ & c^T x \\ & Ax \ge b \end{aligned}}$ $\leftarrow$

$L = \langle A \rangle + \langle b \rangle + \langle c \rangle$

length of input

$K = \{Ax \ge b\}$

$\boxed{\ c\ }$

$\begin{bmatrix} A \end{bmatrix} \begin{pmatrix} x \end{pmatrix} \ge \begin{bmatrix} b \end{bmatrix}$

Ellipsoid algo can return a vertex soln x^* in time poly(L)

$\boxed{LP \in \mathbb{P}!}$

Éva Tardos

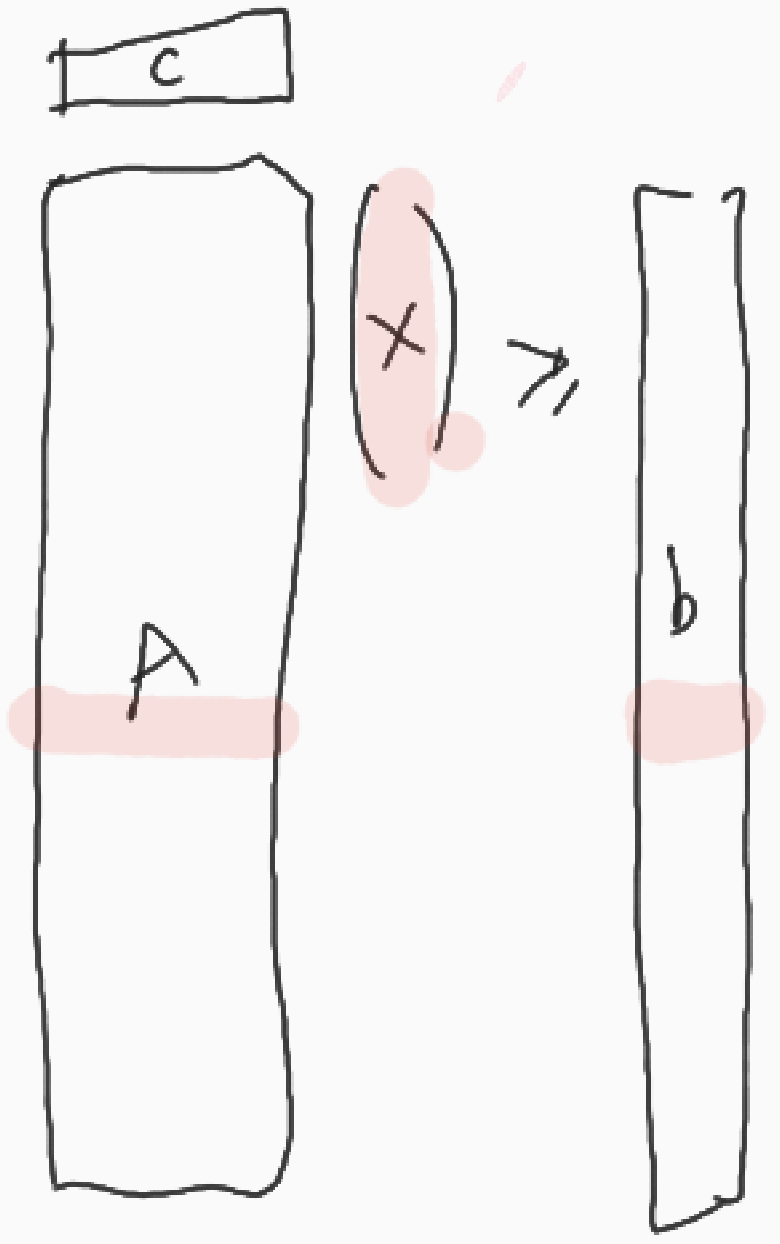
Thm 2: Given LP:

$$\boxed{\begin{array}{l} \min \ c^T x \\ \quad Ax \geq b \end{array}}$$

m constraints
n variables

$$\boxed{L' = \max_{i \in [m]} \langle a_i \rangle + \langle b_i \rangle + \langle c \rangle}$$

$\geq n$



(+ strong separation oracle for A)

find a violated constraint for A

then Ellipsoid algo returns a vertex soln of this LP

in time $\text{poly}(L') \times \text{poly}(n)$

$$\boxed{\text{separation} \iff \text{optimization}}$$

min weight perfect matching in general graphs

$G = (V, E)$ $\quad w_e$ weights

M perfect matchings,

① min wt PM problem

combinatorial algo. $\in P$.

$$\min \sum w_e x_e$$

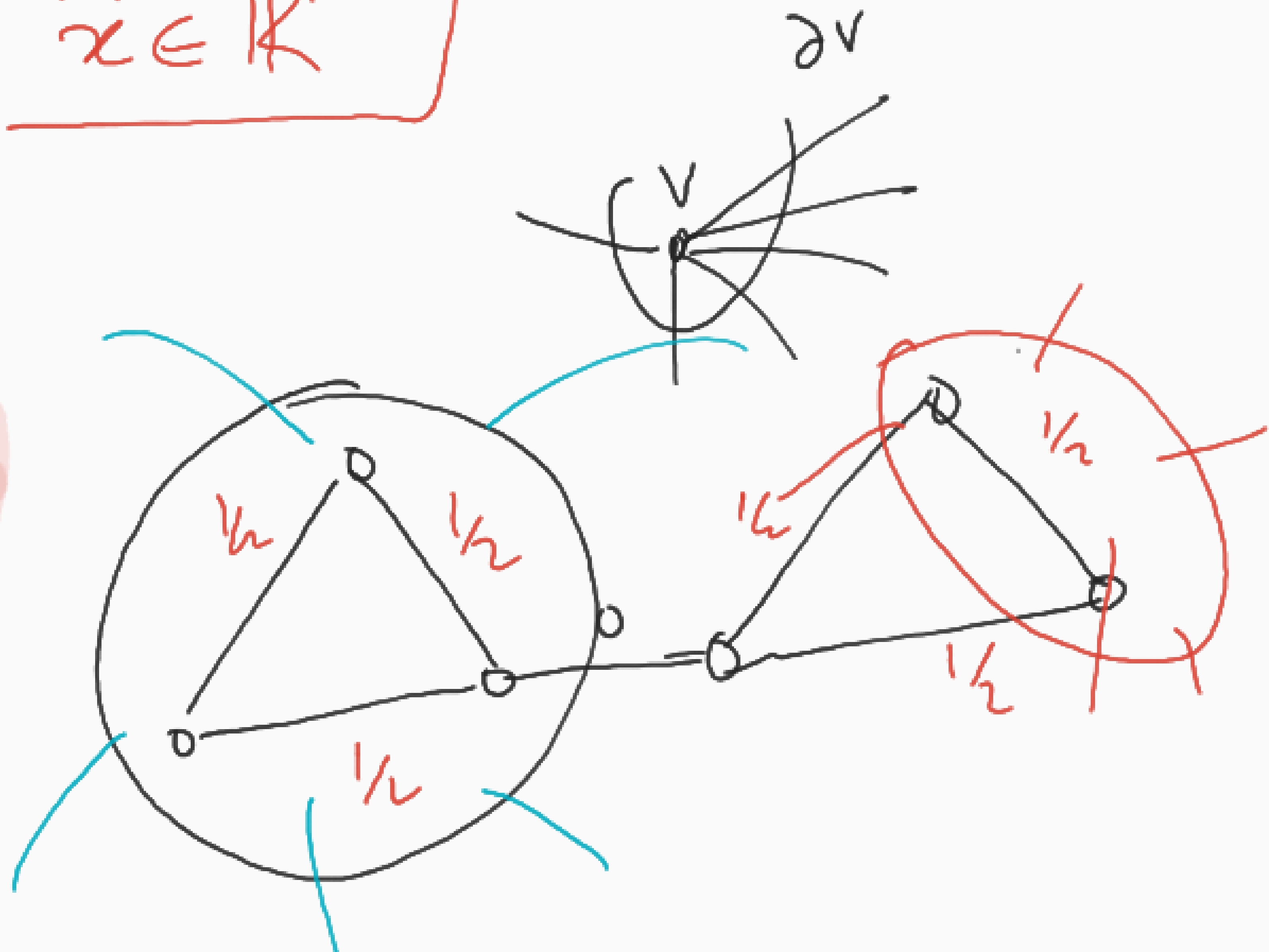
$$\text{st} \quad \sum_{e \in \partial v} x_e = 1 \quad \forall v \in V$$

$$\sum_{e \in \partial S} x_e = x(\partial S) \geq 1 \quad \forall \text{ set } S \subseteq V$$

S odd

$$x_e \geq 0$$

$$\hat{x} \in \mathbb{R}^{|E|}$$



max s-t flow.

directed Graph $G = (V, E)$

s, t

$\mathcal{P}$ = set of all s-t paths.

$$\boxed{\begin{aligned} \max \quad & \sum_{P \in \mathcal{P}} f_P \\ \text{s.t.} \quad & \sum_{P \in \mathcal{P}:\, P \ni e} f_P \le u_e \quad \forall e \\ & f_P \ge 0 \end{aligned}}$$

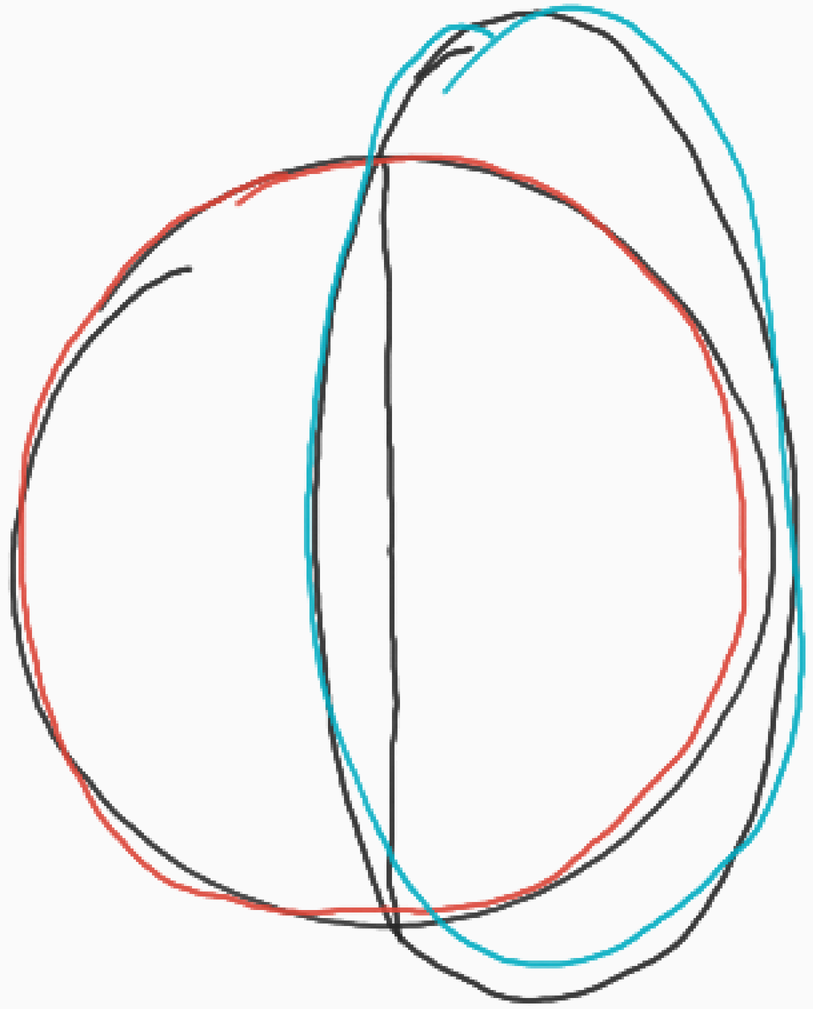
$$\begin{array}{c} 1 \\ \left[\; A \;\right] \end{array} \begin{bmatrix} f \end{bmatrix} \le \begin{bmatrix} u \end{bmatrix}$$

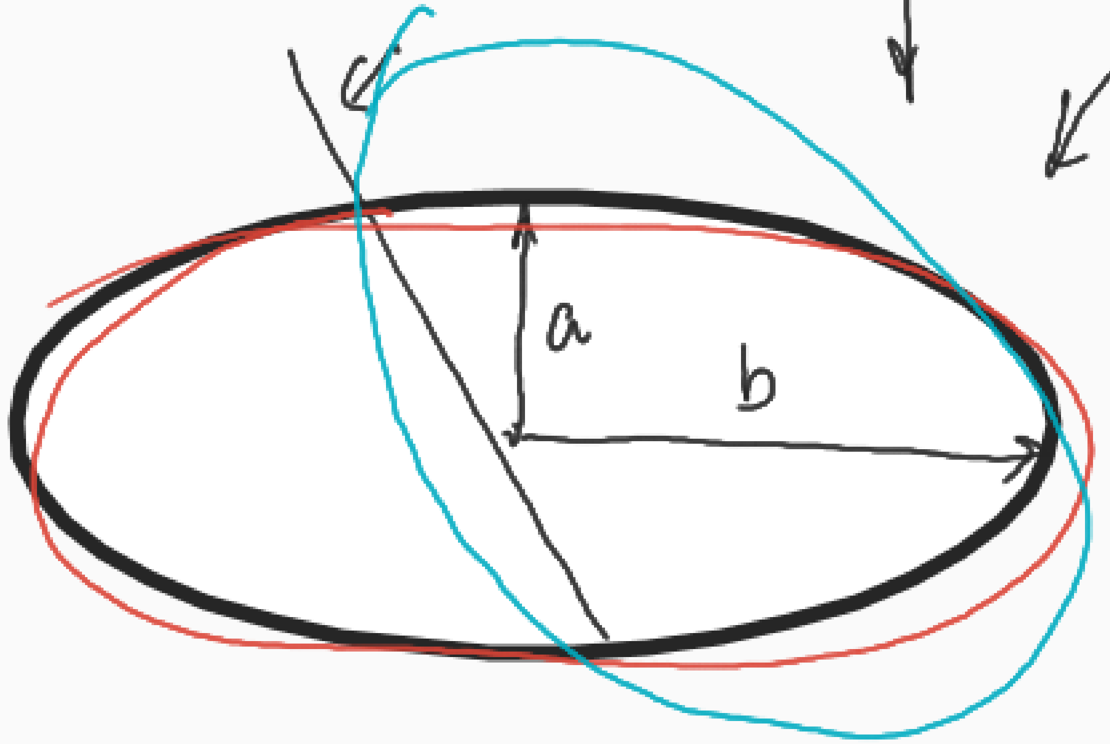
$$\begin{array}{c} u \\ \left[\; A^T \;\right] \end{array} \begin{bmatrix} y \end{bmatrix} \ge \begin{bmatrix} c \end{bmatrix}$$

duality

duality

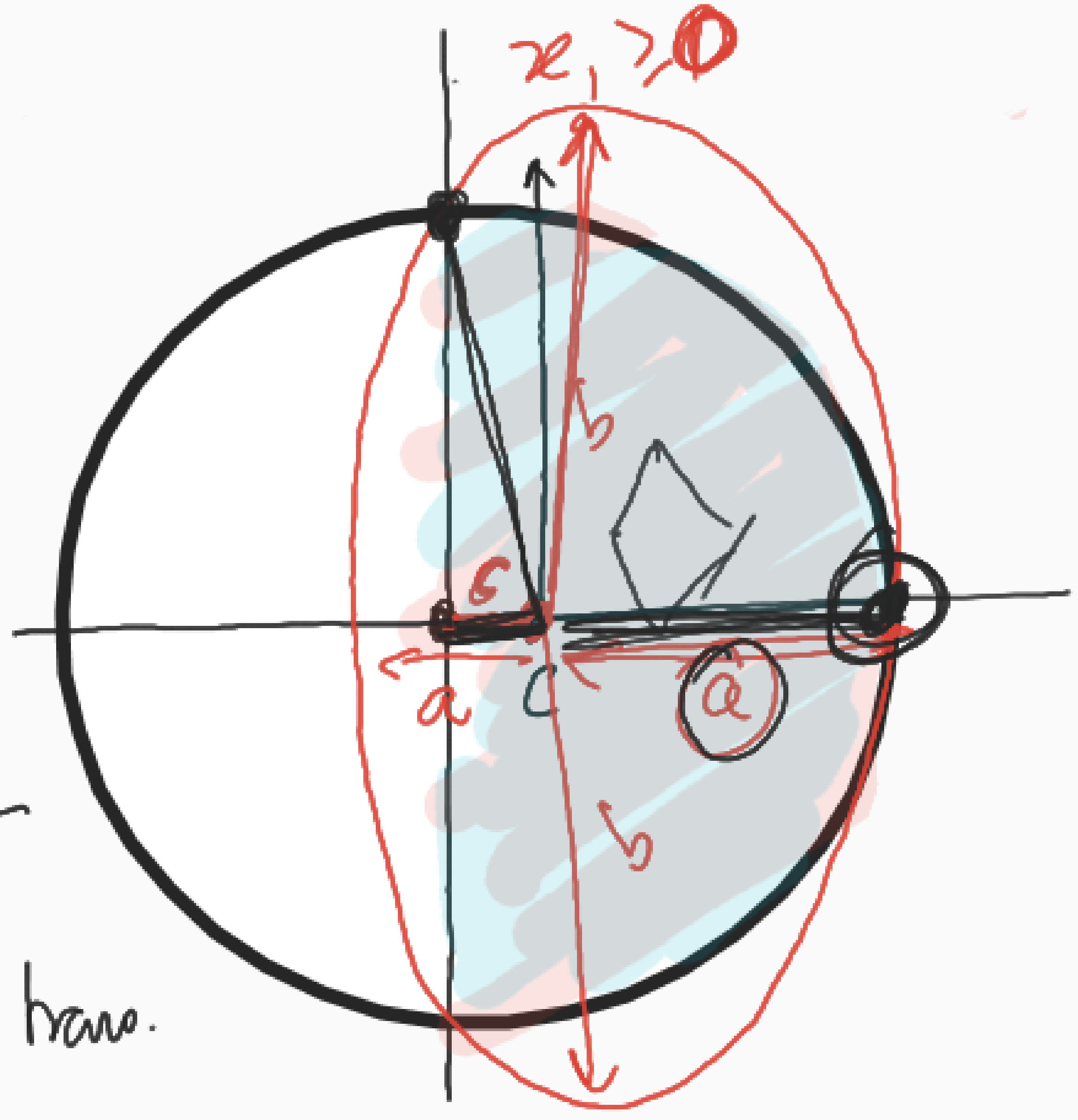
Ellipsoids:





$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$$

Lin trans.



$a . b^{n-1}$

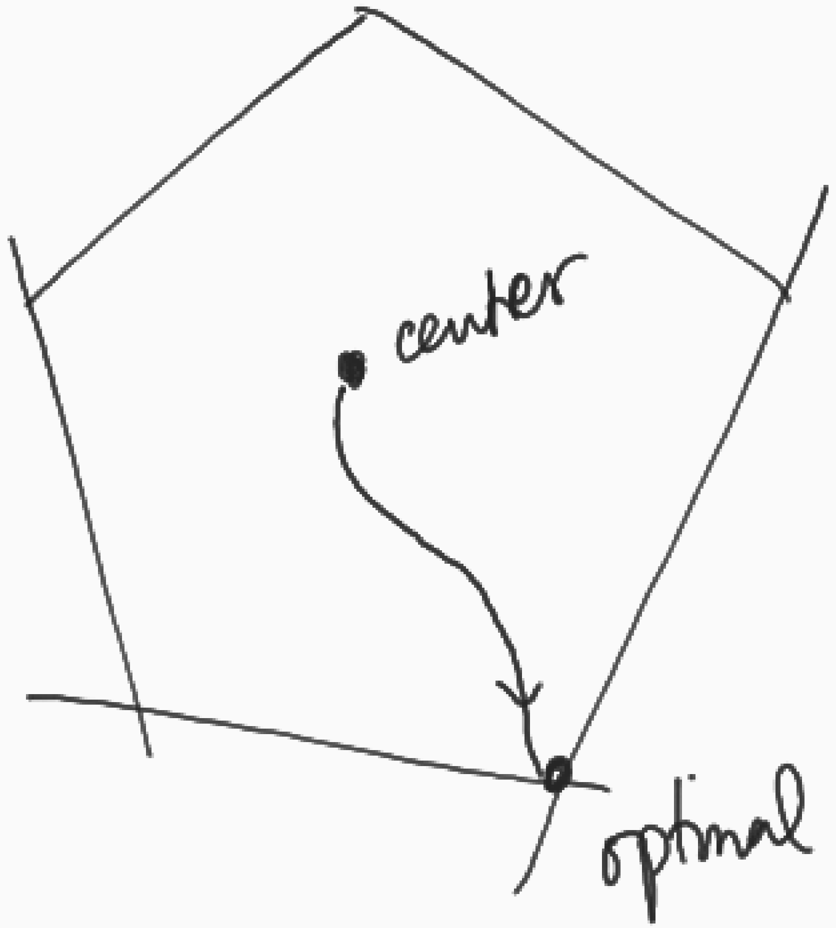
$c + a \geq 1$

$a = 1 - c$

$$\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}\ \cdots\ +\frac{x_n^2}{a_n^2}\le 1$$

- Centroid 2005 ?
- Ellipsoid 1979
- Simplex 5? ←
- Geometric ← MW, Random (Seidel) 451
- Interior Point Methods 84 Karmarkar

  Path Following IPM



$x_1 \ldots x_n$

$$\min \; c^T x$$

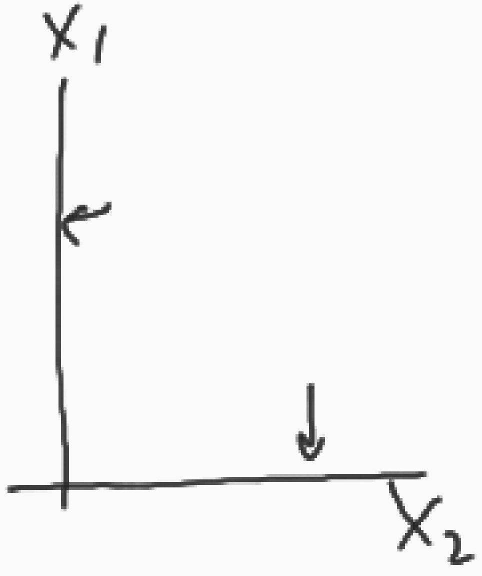
$$[\,Ax=b\,]$$

$$x \geq 0$$

Barrier term

$$\min f_\eta(x) = c^T x + \eta \left(\sum_{i=1}^{n} \log \frac{1}{x_i} \right)$$

$$Ax = b$$



Ay

$\eta_0 \qquad x_0 = \hat{x}_0$

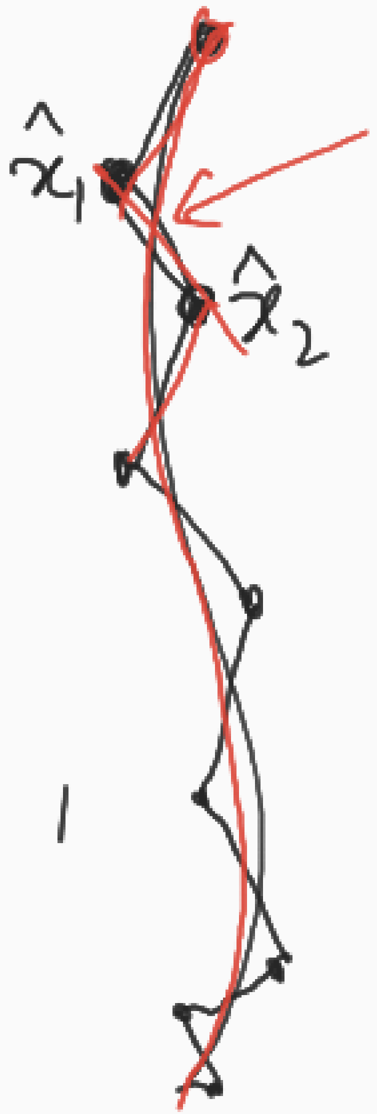
$\eta_0\left(1-\frac{1}{\sqrt{n}}\right) = \eta_1 \qquad x_1 \qquad \hat{x}_1$

$\eta_1\left(1-\frac{1}{\sqrt{n}}\right) \qquad \eta_2$

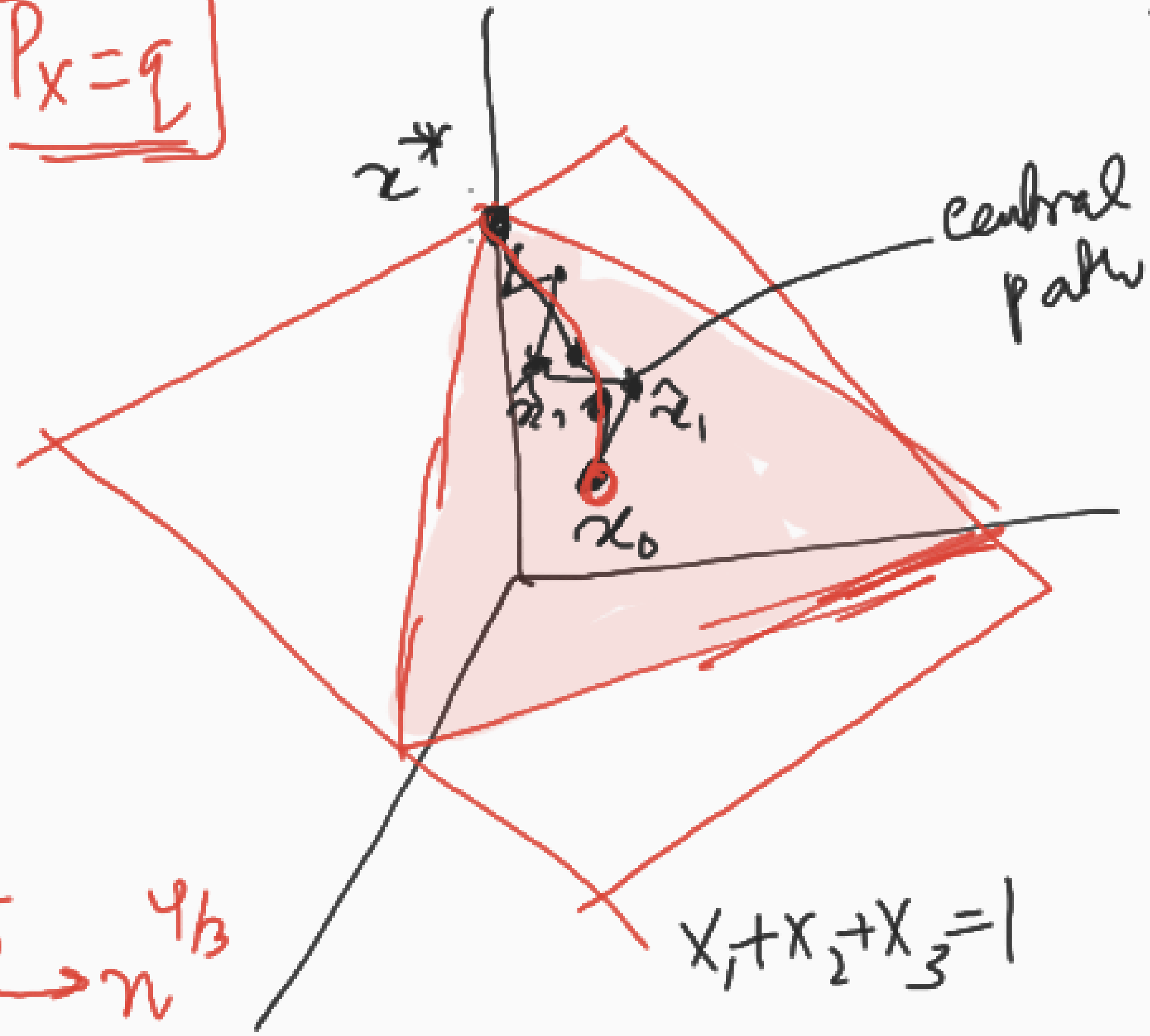
$$\boxed{\sqrt{n}}\, \log\left(\frac{n\eta_0}{\varepsilon}\right)$$

linear system

Solve $\boxed{Px=q}$



ε-close



$n^{1.5} \rightarrow n^{4/3}$