Lecture 20: Centroids and Ellipsoids (Dimension-Dependent)

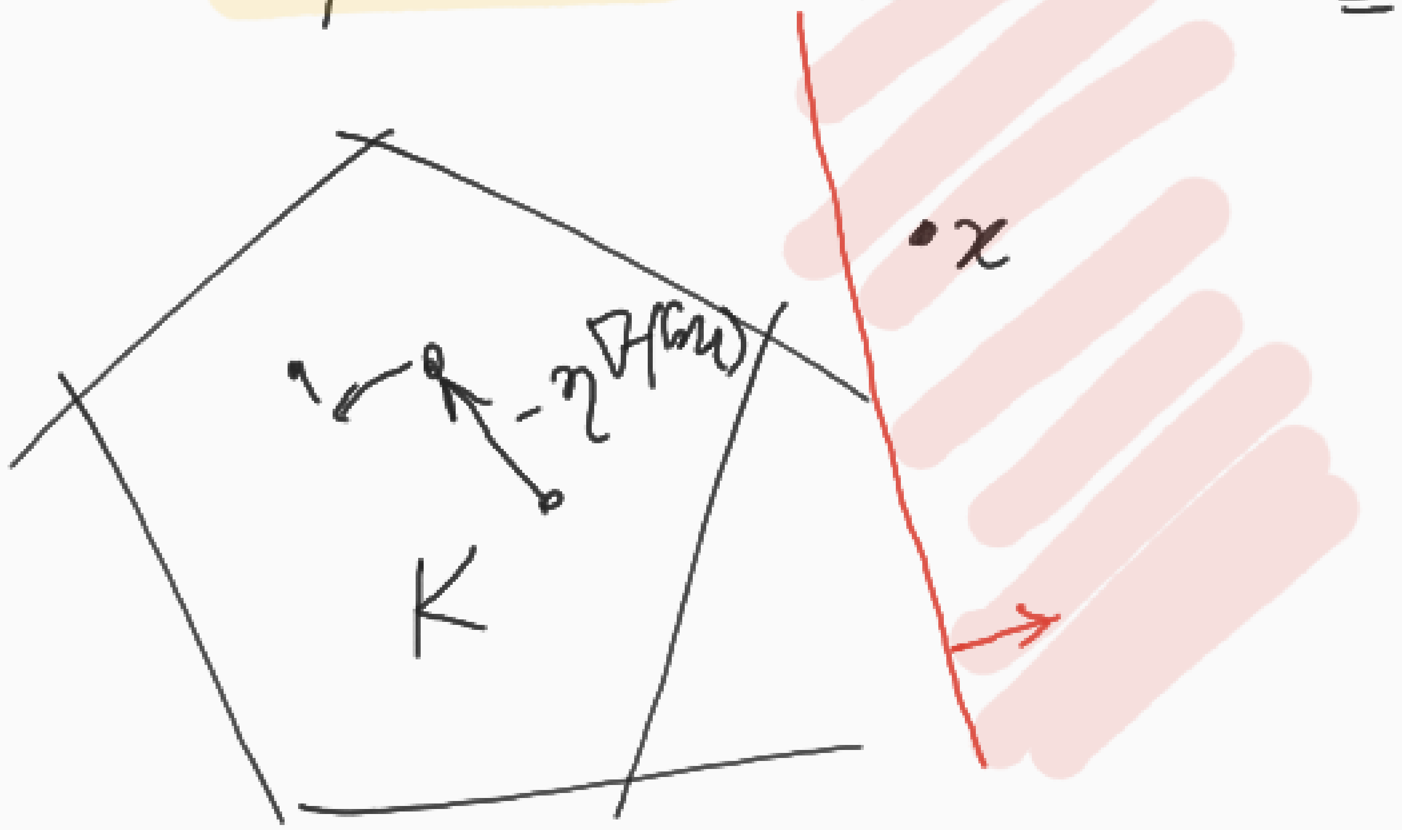
- centroid algo
- Ellipsoid algo
  - Statements
  - strong separation oracles
  - some details of the algo.

convex / linear

$\min f(x)$ s.t. $x \in K$

$K = \{x : Ax \le b\}$

- Feasibility (K) is $K \neq \phi$? if so, return $x \in K$
- Separation is $x \in K$?
- Optimization $f(x) \le f(x^*) + \varepsilon$

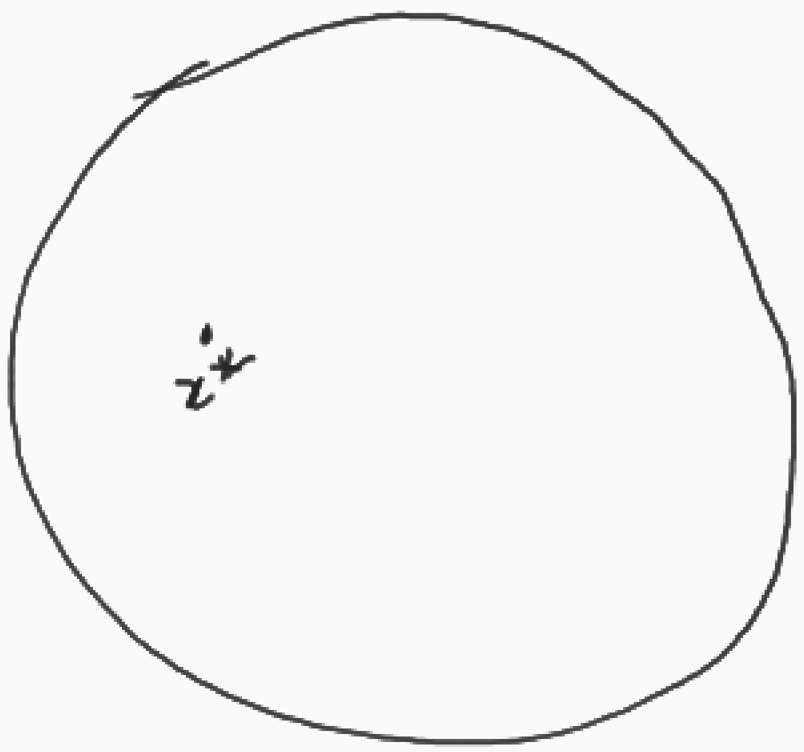


Centroid Algo: $f(x) \le f(x^*) + \varepsilon$ $\quad \log(1/\varepsilon)$ "iteration"

$\boxed{x \in K}$

$x^* = \arg\min_{x \in K} f(x)$

Assume given any K', compute $\begin{cases} \text{center-of gravity} \\ \text{centroid} \\ \text{barycenter} \end{cases} (K)$

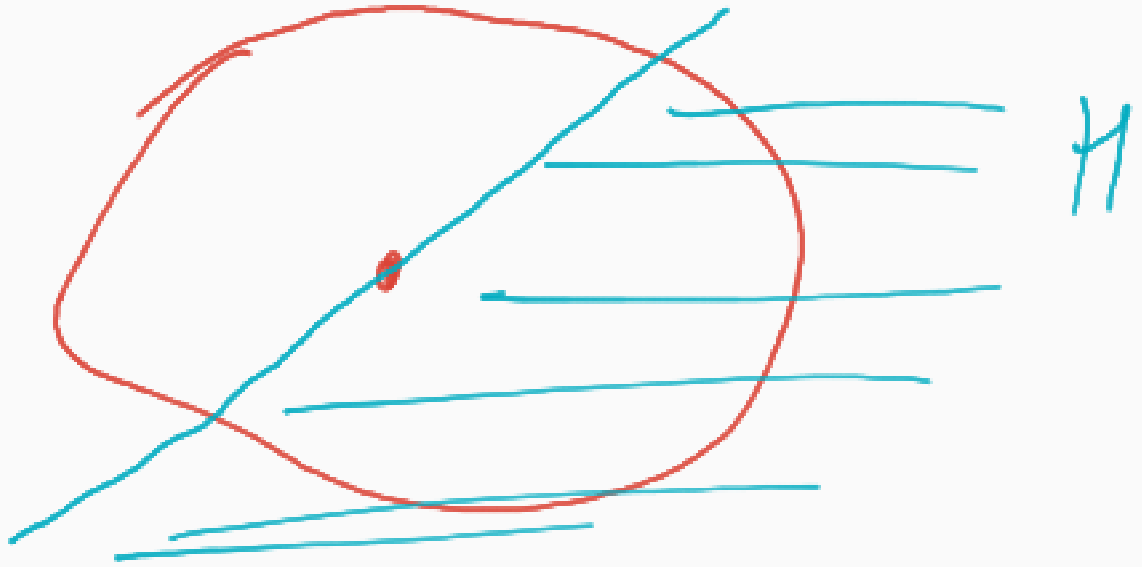


$S = \bar{x}_1, x_2 \ldots x_N \in \mathbb{R}^n$ $\quad \boxed{cg(S) = \frac{\sum_i \bar{x}_i}{n}}$

$cg(K) = \frac{\int_{x \in K} x \, dx}{vol(K)}$ "centroid"

Grunbaum's Thm

$\frac{1}{e} \le \frac{vol(K \cap H)}{vol(K)} \le 1 - \frac{1}{e}$



$\min f(x) \leftarrow$ convex fn

$x \in B_{(0,1)} = K_1$

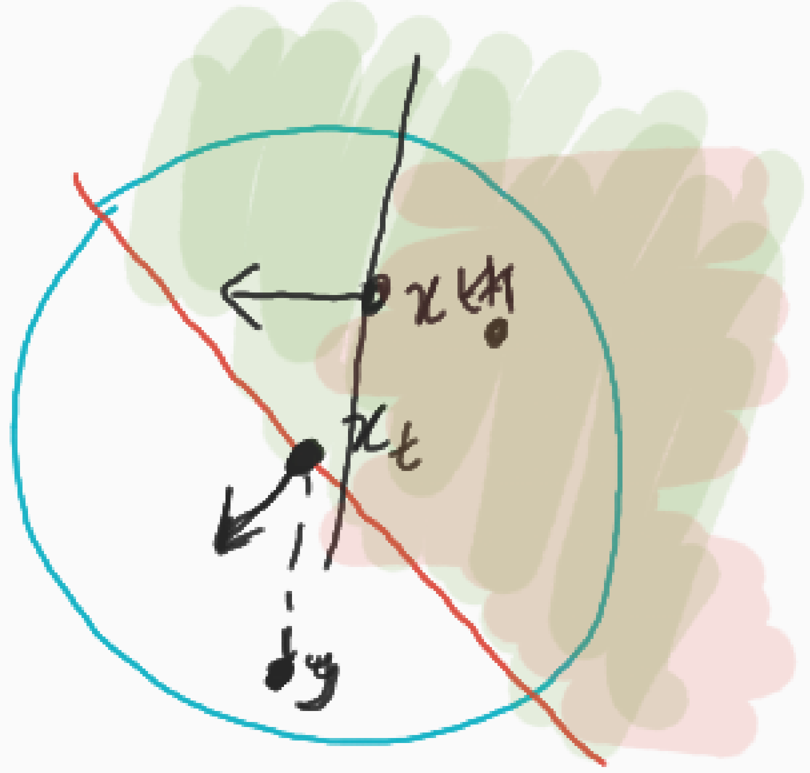
$\underline{x_t} = cg(K_t)$

compute$(\nabla f(x_t))$

$$K_{t+1} = K_t \cap H_t = \{x \mid \langle \nabla f(x_t), x - x_t \rangle \le 0\}$$

stop after T steps

$x_1, x_2 \;\dots\; x_T$ return $\hat{x} = \operatorname{argmin}_{t=1}^{T} f(x_t)$

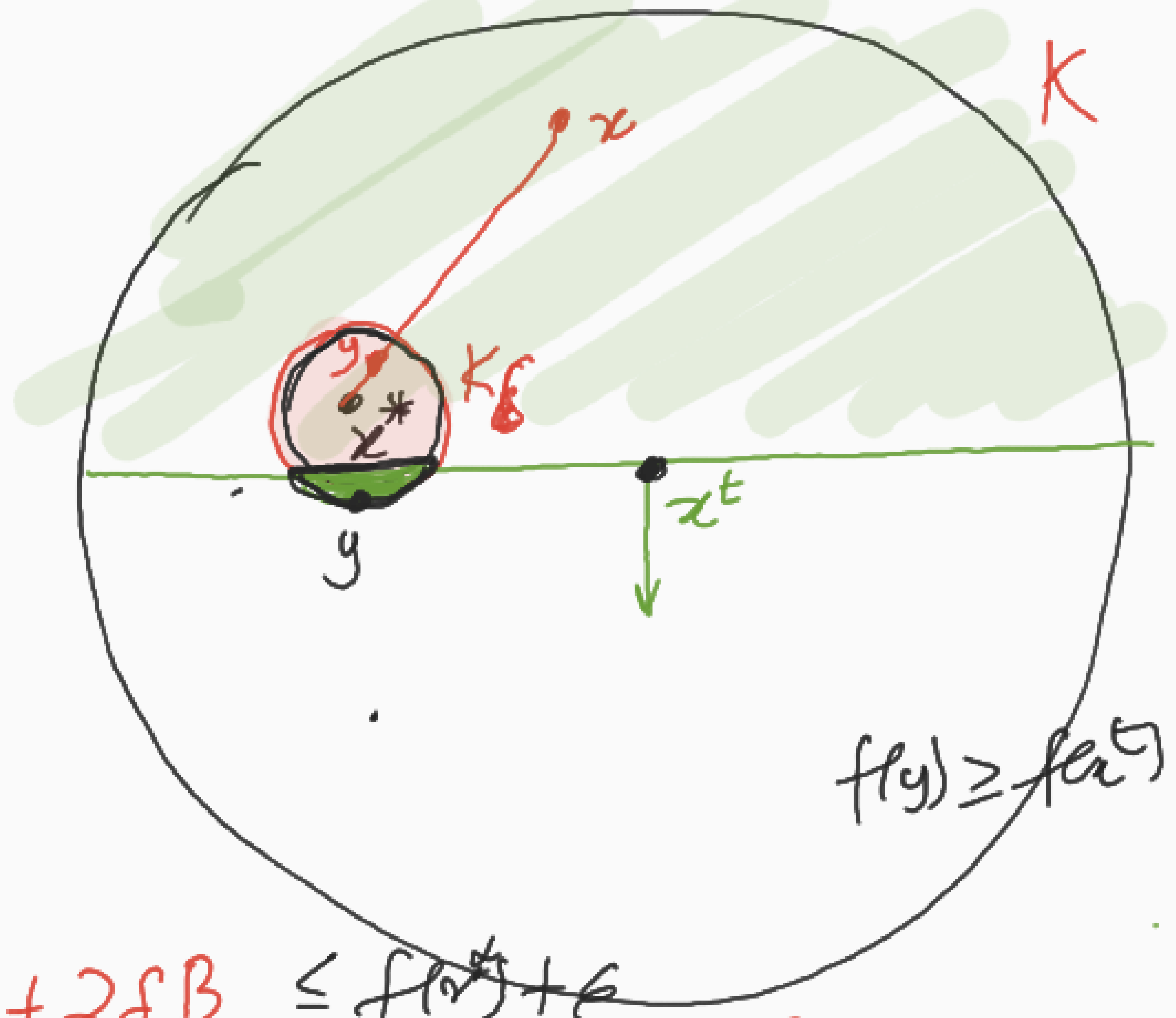


$$\boxed{f(y) \ge f(x_t) + \underbrace{\langle \nabla f(x_t), y - x_t \rangle}_{\ge 0}}$$

Thm Suppose $f: K \to [-B, B]$

then $f(\hat{x}) \le f(x^*) + \varepsilon$

after $T = 3n \ln\left(\frac{4B}{\varepsilon}\right)$ steps



$K_\delta = \{(1-\delta)x^* + \delta x \mid x \in K\}$ δ-shrunk copy of K.

Fact 1: $\mathrm{vol}(K_\delta) = \delta^n \cdot \mathrm{vol}(K)$

Fact 2 value of f on $y \in K_\delta$ is $f(y) \le f(x^*) + 2\delta B \le f(x^*) + \varepsilon$

$$f(y) = f((1-\delta)x^* + \delta z) \le (1-\delta) f(x^*) + \delta \overbrace{f(z)}^{\le B}$$

$$= f(x^*) - \delta f(x^*) + \delta f(z) \quad f(x^*) \ge -B, \; f(z) \le B$$

$$\le f(x^*) + 2\delta B.$$

Fact 3

$\delta = \frac{\varepsilon}{2B}$

Fact 3: if $B^\delta \subseteq K_t$ $B^\delta \not\subseteq K_{t+1}$ $\Rightarrow \exists$ pt $y \in B^\delta \setminus K_{t+1}$

st $f(x_t) \le f(y)$

$f(\hat{x}) \le$ $\le f(x^*) + \varepsilon$

$\Rightarrow$

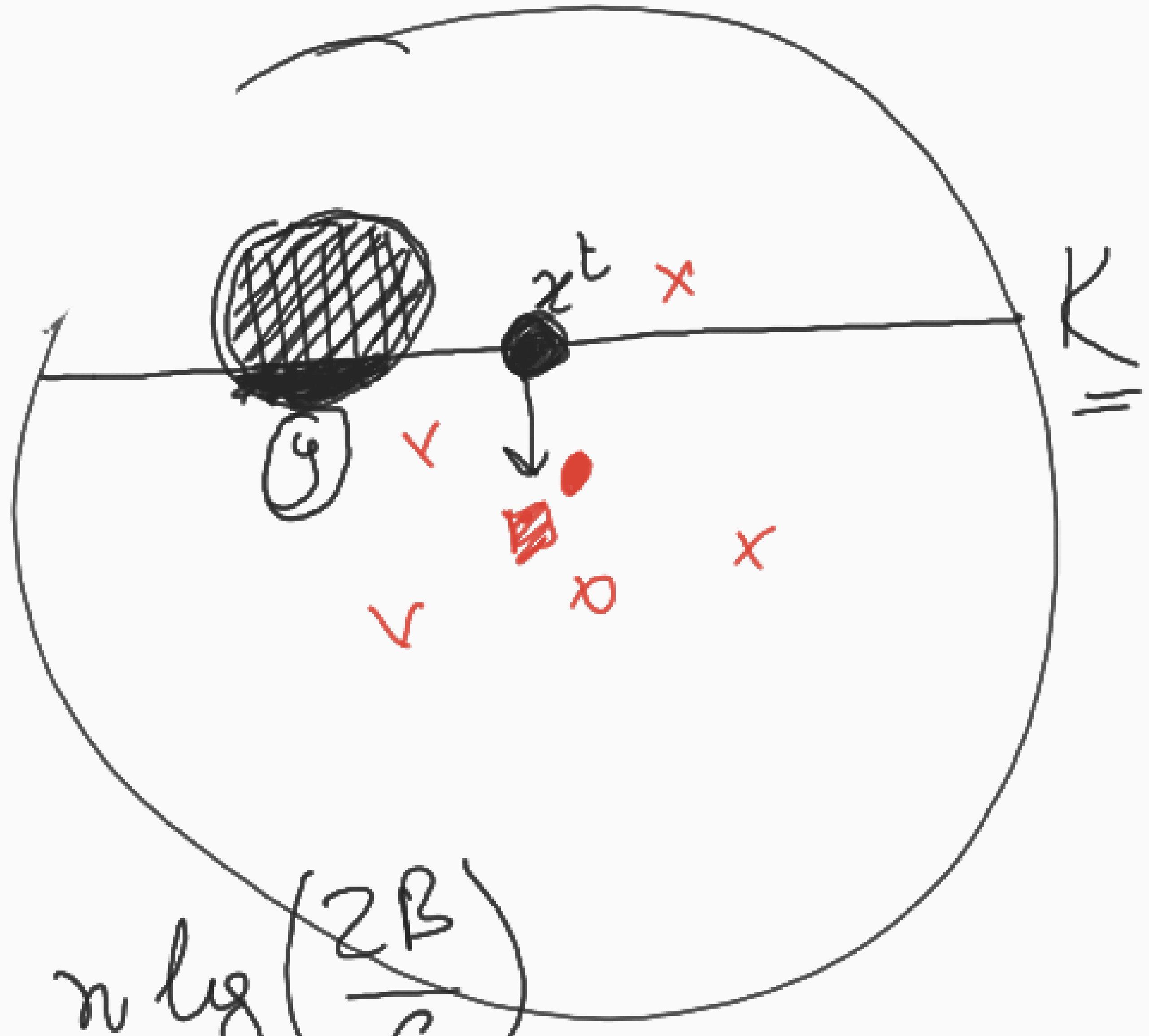
$$\boxed{vol(K^\delta) = \delta^n \cdot vol(K)}$$

$$\Rightarrow vol(K_T) \le vol(K_1) \cdot \left(1 - \tfrac{1}{e}\right)^T$$

$$\approx vol(K_1) \left(\tfrac{1}{2}\right)^T$$

$$\le vol(K_1) \cdot (\delta/2)^n.$$

if $T = n \lg\left(\frac{2}{\delta}\right) = n \lg\left(\frac{2B}{\varepsilon}\right)$



Dyer Frieze
Kannan

centroid #P-hard

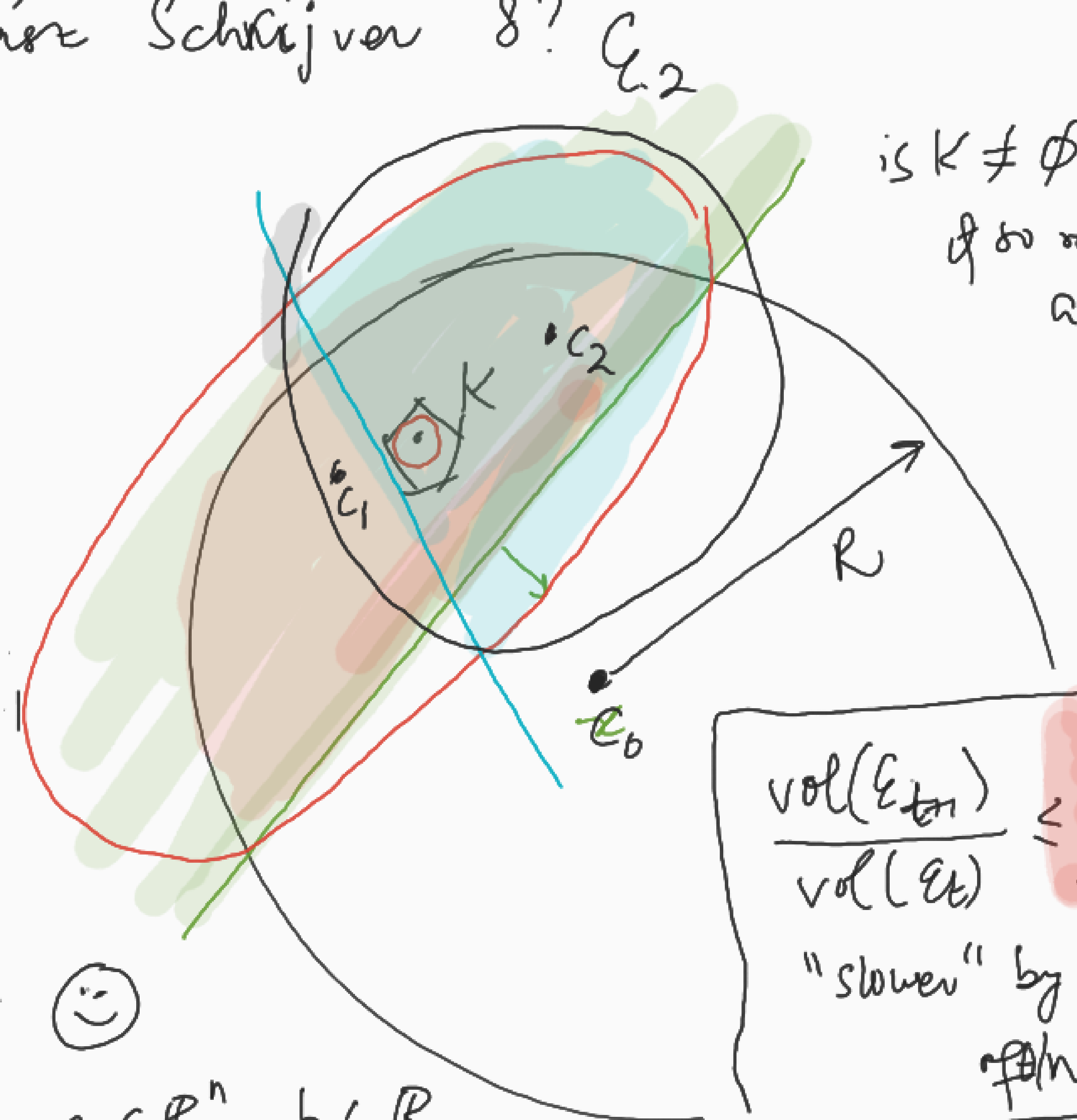
Ellipsoid Algo: Grotschel Lovász Schrijver 8?

$K \subseteq B(0,R)$

$B(c,r) \subseteq K$

$\lg \frac{R^n}{r^n}$

Given: K, R, r — find $x \in K$ — $\boxed{2(n+1)n \lg \frac{R}{r}}$



is $K \neq \emptyset$?
if so return a pt $x \in K$

$$\frac{vol(\mathcal{E}_{t+1})}{vol(\mathcal{E}_t)} \leq e^{-\frac{1}{2(n+1)}}$$

"slower" by factor $f(n)$.

Strong Separation Oracle for K:

Given x, if $x \in K \Rightarrow$ "Yes" ☺

$x \notin K$, returns $a \in \mathbb{R}^n$, $b \in \mathbb{R}$
st $a^T x > b$ but $a^T y \leq b$ for $\forall y \in K$

N. Shor Simplex Dantzig 54?

1979 Khachiyan LPs in poly time using Ellipsoid

Thm. Given LP $\max\{c^T x \mid Ax \le b\}$, m constraints, n variables.

$\mathbb{Q}$ $\langle A \rangle = \sum_{ij} \langle a_{ij} \rangle$ $\quad \langle A \rangle + \langle b \rangle + \langle c \rangle$

Can find a vertex soln in time $\boxed{\mathrm{poly}(\underline{\langle A \rangle} + \underline{\langle b \rangle} + \underline{\langle c \rangle})}$
or report infeasible / unbounded.

Q: find algo s.t. # of arithmetic ops (over #s of size
is poly$(m\,n)$. ?

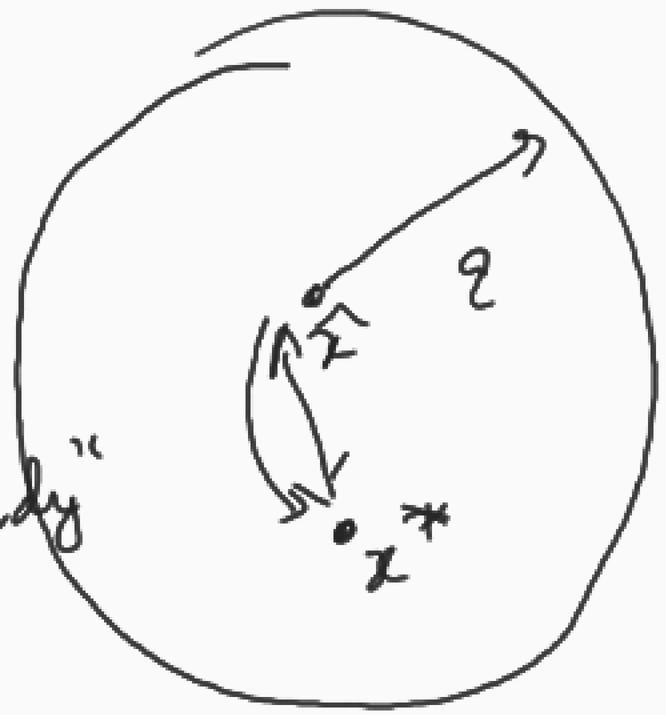
$\boxed{\text{Strongly Poly LP algo}}$

max $c^T x$

$Ax \le b$

assuming feasible & finite

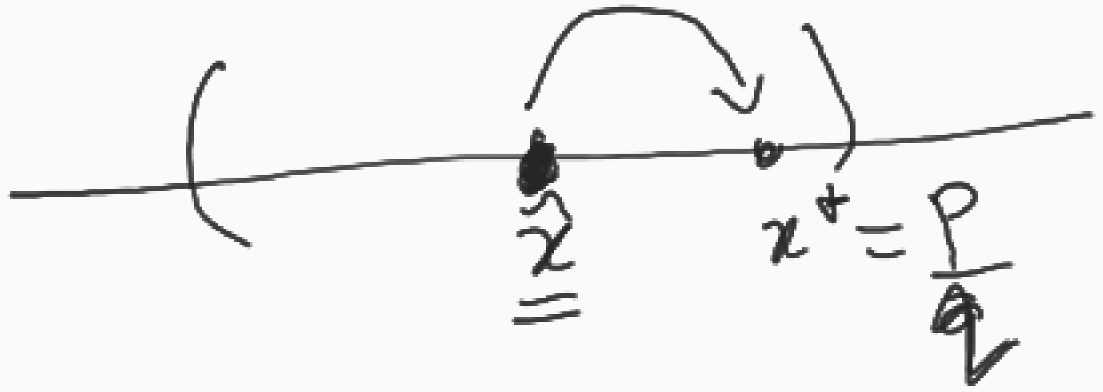
① find upper & lower bounds on $c^T x$ $\quad [LB, UB]$ "rounding"



$$\boxed{\begin{array}{c} z \le c^T x \le z + \varepsilon \\ Ax \le b \end{array}} \quad K_z$$

Binary / Doubling search

find $\hat{x}$ st $\boxed{\begin{array}{c} c^T \hat{x} \le c^T x^* + \varepsilon \\ A\hat{x} \le b \end{array}}$



q has ≤ 1 digit

Ex: show that $\exists\, x^*$ (vertex sol^n) st. # bits in numerators, denom $\le b$

$$K = \{ Ax \le b \}$$

$$K + B_\varepsilon = K_\varepsilon$$

$$\begin{cases} \hat{x} \in K_\varepsilon \\ c^T \hat{x} \le c^T x^* + \varepsilon \end{cases}$$

$$\Downarrow$$

round?