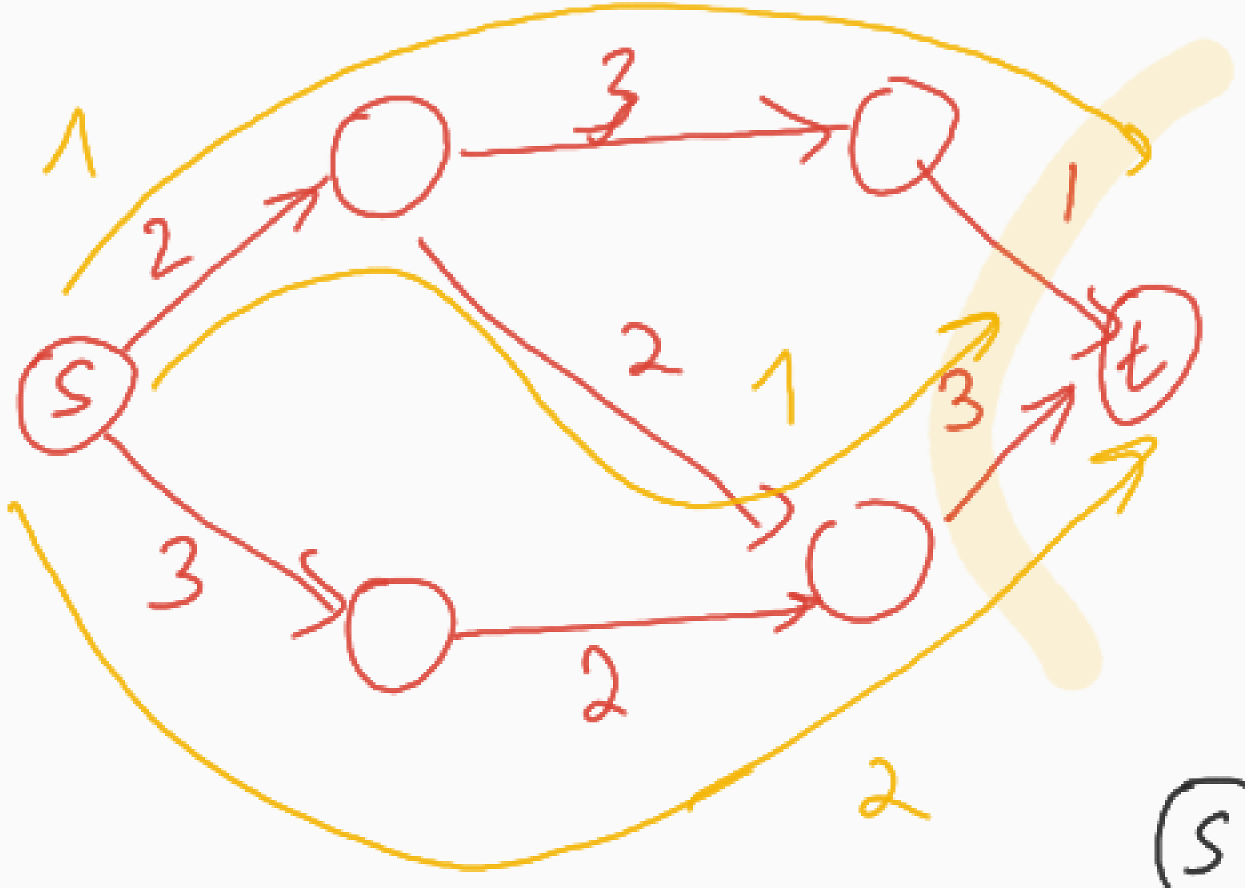
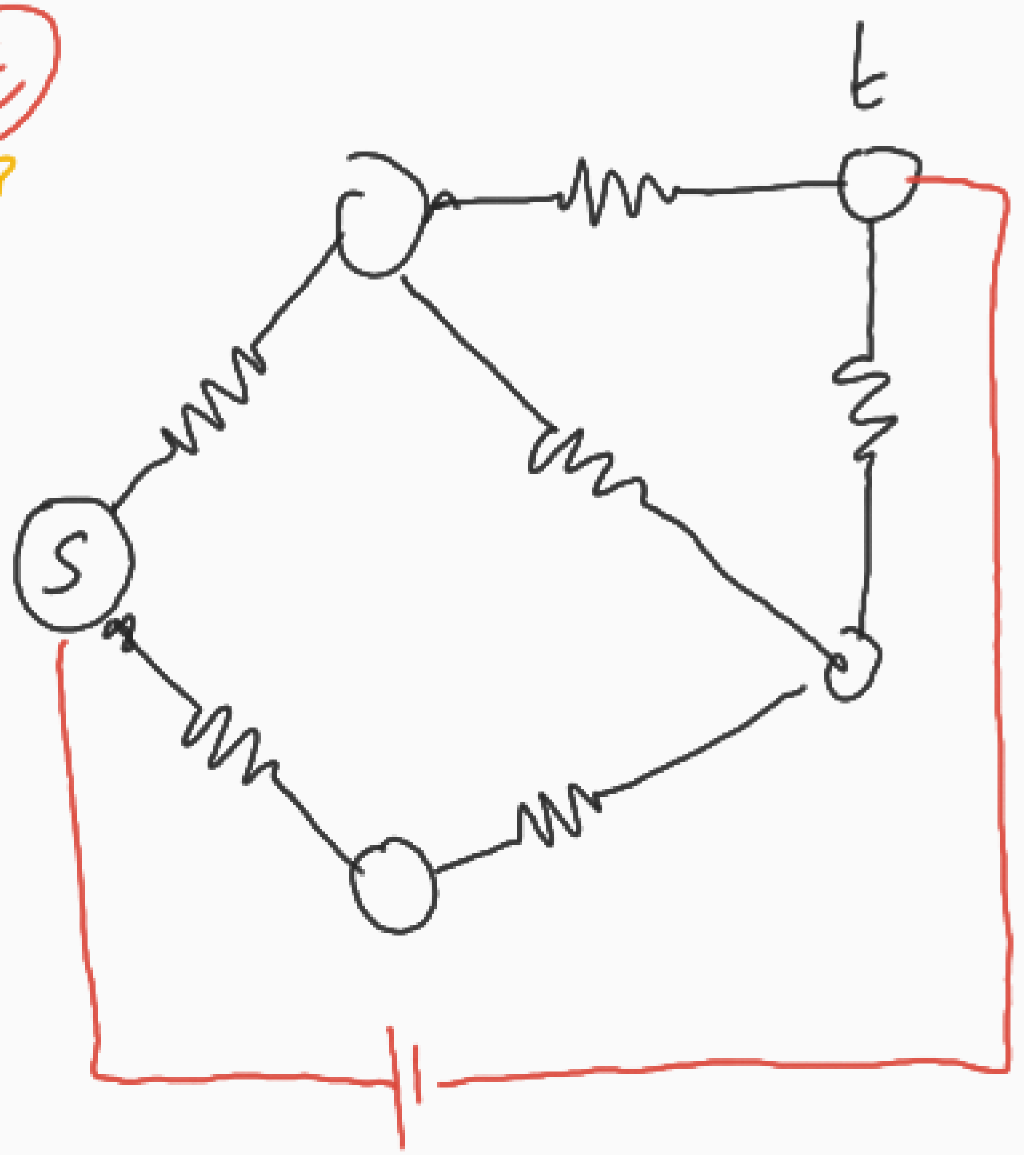
Lecture 16: Solving LPs





- Recap of LP solving
  - s-t flows
  - better results
  - Electrical flows.

$(1+\varepsilon)$

Thm: apx max s-t flow algo

runs in time $O(\underline{\underline{m}}^{4/3} \text{polylog} / \text{poly}(\varepsilon))$

undirected, unweighted

↑ ↑ not essential

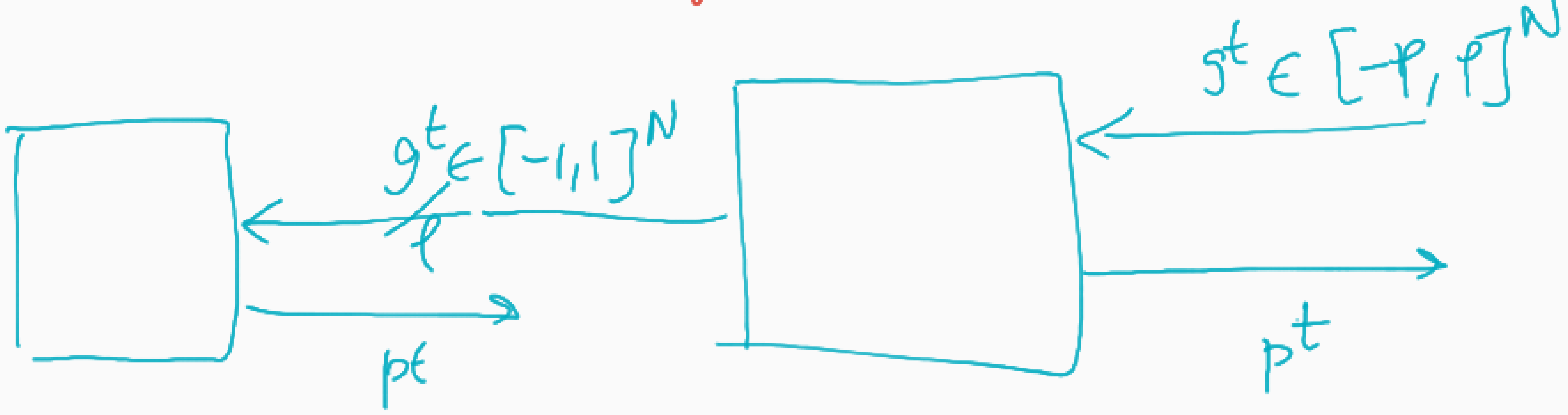
<u>Hedge</u>: @ each time Algo plays $p^t \in \Delta_N$

Adversary plays $g^t \in [-\rho, \rho]^N$

gain to Algo $= \langle g^t, p^t \rangle$

<u>Thm</u>: $\varepsilon \le 1/2$. Then $\underbrace{\frac{1}{T}\sum_{t=1}^{T} \langle g^t, p^t \rangle}_{\text{avg gain}} \geq \underbrace{\max_i \frac{1}{T}\sum_{t=1}^{T} \langle g^t, e^i \rangle}_{\text{best avg gain for any expert}} - \frac{\varepsilon T}{T}$

$\boxed{\text{if } T \geq \frac{4 \rho^2 \log N}{\varepsilon^2}}$



$$\boxed{\begin{array}{l} \cancel{\max}\ \boxed{\langle c, x\rangle = OPT} \\ Ax \le b \\ x \ge 0 \end{array}}$$

$$\boxed{\begin{array}{l} \cancel{\max}\ \langle c, x\rangle = OPT \\ \alpha^T x \le \beta \\ x \ge 0 \end{array}}$$

single constraint

$$\begin{array}{c} p^t \\ 0.1 \\ 0.7 \\ 0.05 \\ 0.1 \\ 0.05 \end{array} \begin{bmatrix} \\ & A & \\ \\ \end{bmatrix} \begin{pmatrix} x \end{pmatrix} \le \begin{bmatrix} b \end{bmatrix}$$

"Easy" polytope : $K = \{x : \langle c, x\rangle = OPT,\ x \ge 0\}$

Oracle : solve for any pt $x \in \boxed{K \cap \{x : \alpha^T x \le \beta\}}$

$T = \#\text{ rounds} = \dfrac{4\rho^2 \log m}{\varepsilon^2}$

Algo: at each time $t \in 1 \dots T$ =

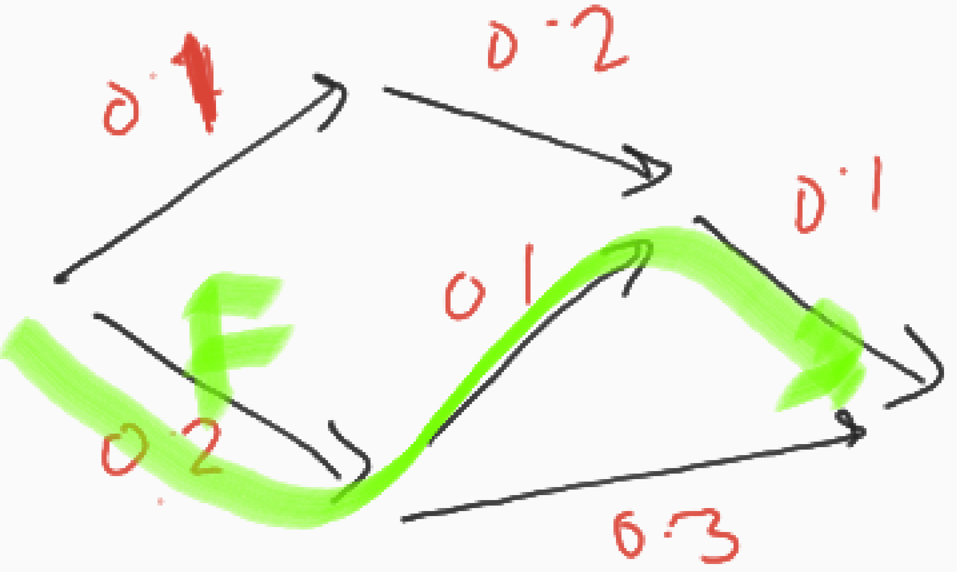
combine the constraints in $Ax \le b$ using weights $p^t \Rightarrow \langle \alpha^t, x\rangle \le \beta^t$

use oracle to solve, get x^t

gain of constraint i = violation $(Ax^t)_i - b_i$ ← ρ: "width": absolute max value of this violation

F-flow

~~Max~~ s-t flow problem: (directed, unit cap edges)
m, n



$\mathcal{P}$ = set of all s-t paths

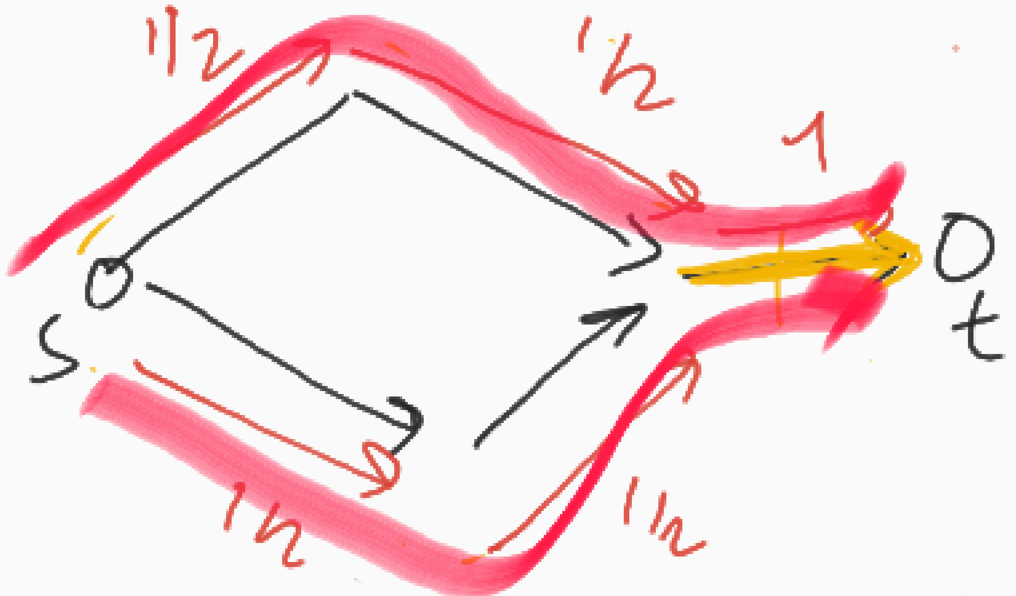
• $\{f_p\}_{p \in \mathcal{P}}$ exponential #

$$\sum_{p \in \mathcal{P}} f_p = F$$

$$m \quad \sum_{p : e \in p} f_p \le cap(e) = 1 \quad \forall e$$

$$f_p \ge 0$$

flow : $\rightleftarrows$ path decomposition



$\longrightarrow K = \{ f_p \ge 0, \ \sum f_p = F \}$

$f \in K$ st $\sum f_p \cdot len(p) \le 1$

$$\sum_p f_p \cdot len_t(p) = \sum_p f_p \cdot \sum_{e \in p} b_e^t = \sum_e b_e^t \sum_{p : e \in p} f_p \le \sum b_e^t = 1$$

Algo: Hedge $\rightarrow \{p_e^t\}_{e \in E}$

Find the shortest path P wrt edge wts $\boxed{p_e^t}$

$x^t = F$ units of flow on it

give gains to Hedge to update ~~p_e^t~~ values.

// $w_e^{t+1} \leftarrow w_e^t \cdot e^{\varepsilon \cdot \text{gain}} \simeq w_e^t (1 + \varepsilon . \text{violation})$

if LP is feasible

Average constraint

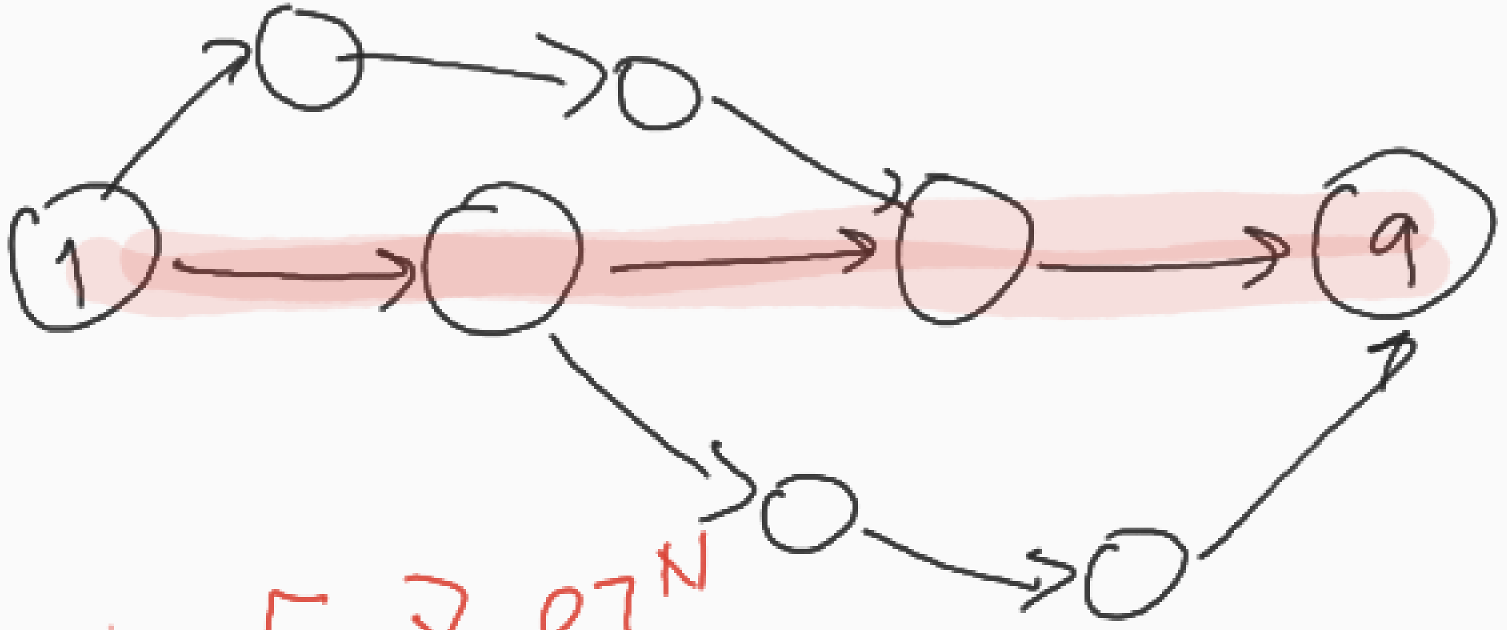
$$\boxed{\sum_p f_p \, \text{load}_t(P) \leq 1}$$

$$\hat{x} = \frac{x^1 + x^2 + \cdots + x^T}{T}$$

guarantee: $A\hat{x} \leq b + \varepsilon \mathbb{1}$

$$\sum_{P: e \in P} \hat{f}_P \leq 1 + \varepsilon \quad \forall e$$

after: $\dfrac{\rho^2 \cdot \log m}{\varepsilon^2}$



gains $[-\gamma, \rho]^N$

width: $\frac{\gamma \rho \lg m}{\varepsilon^2} \le \frac{F \lg m}{\varepsilon^2}$ #iteration

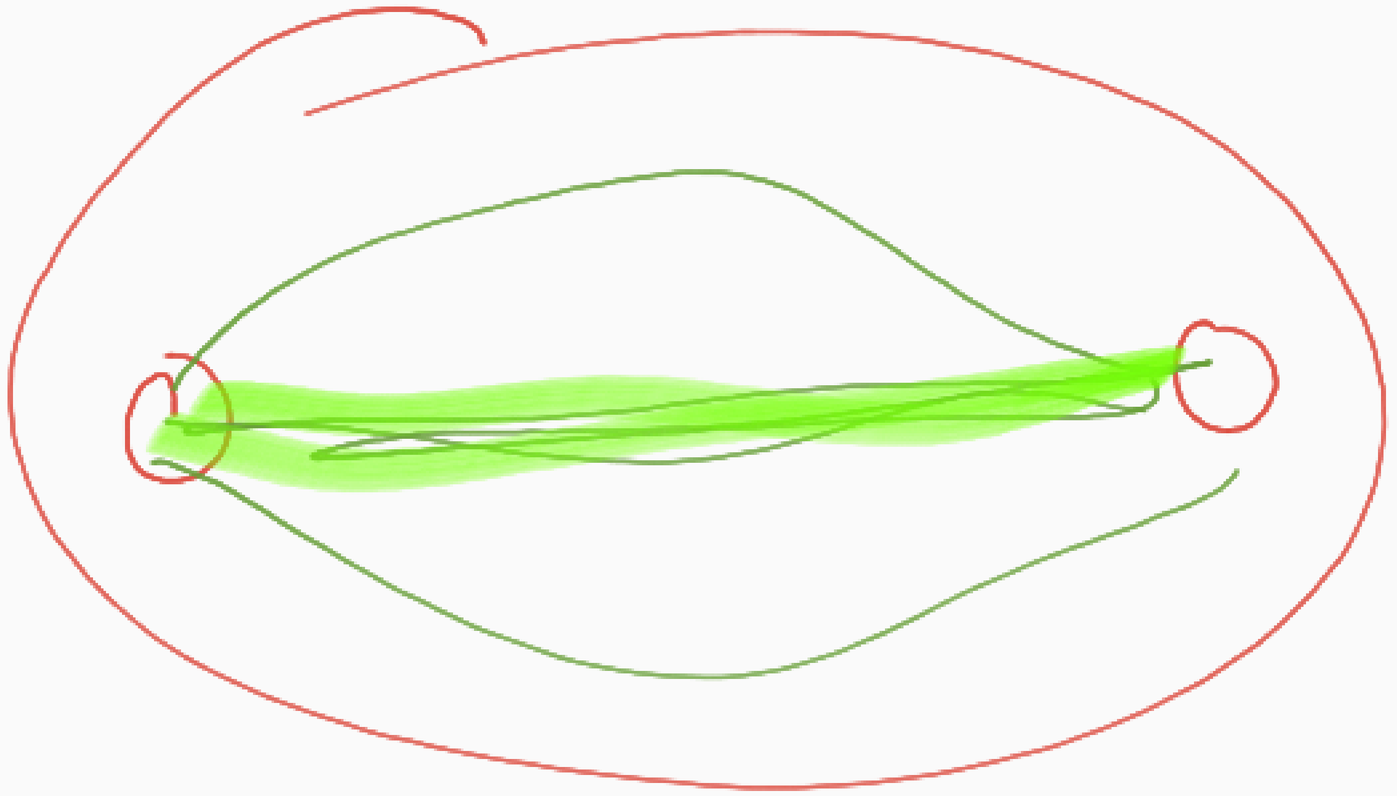
ρ : $F-1$

γ : 1

$\times\, O(m + n \lg n)$

Runtime: $\tilde{O}(mF/\varepsilon^2)$

$O(mF)$ (F-F)



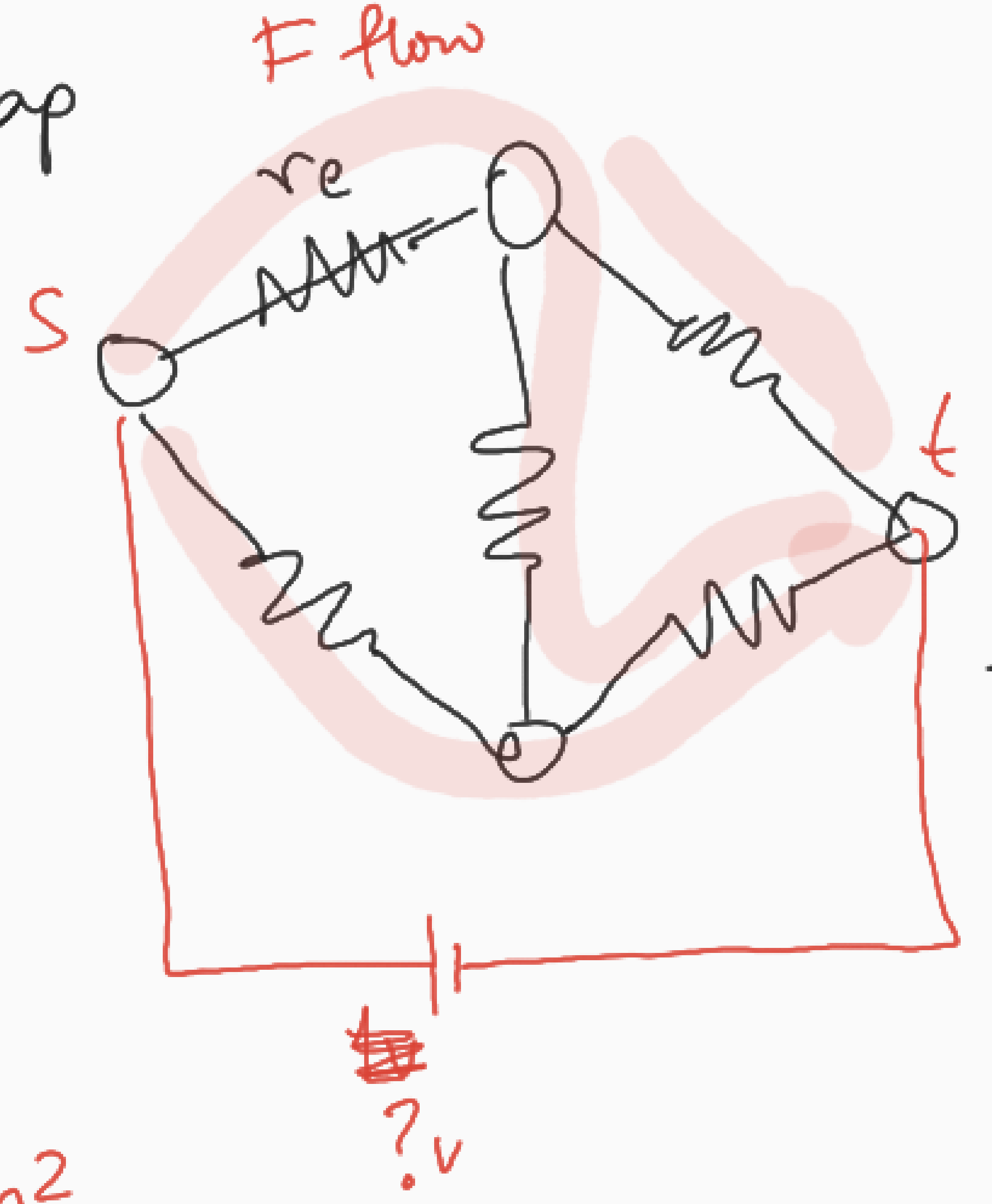
if $\exists$ flow of F

$\Rightarrow \exists$ shortest path of length $\le \frac{1}{F}$

$\Rightarrow$ pushing F units with cost

$\sum f_p \operatorname{len}(P) \le 1$

Max Flows using Electrical Flows (Undirected) unit cap



View graph G as electrical network (resistive)

$r_e = \frac{1}{cap(e)}$

s-t flow minimizes energy burn of value F

$$\underset{\substack{f \text{ flow from} \\ s \text{ to } t \\ \text{of value } F}}{\text{argmin}} \ \underline{\underline{\mathcal{E}(f)}}$$

$$\| \quad \sum_{e\in E} f_e^2 r_e$$

Sps $\exists$ cap. resp flow $\overset{f^*}{}$ $s \to t$ of value F $\qquad \mathcal{E}(f^*) = \sum_e (f_e^*)^2 \le m$

$\Rightarrow$ Elec flow $\mathcal{E}(f) \le m \Rightarrow f_e \le \sqrt{m} \ \forall e$

$\Rightarrow$ width of electrical flow $\leq \sqrt{m}$

$\Rightarrow$ runtime $= O\left(\frac{\sqrt{m}\,\log n}{\varepsilon^2}\right) \underbrace{\boxed{(\text{time to find elect flow})}}_{\tilde{O}(m)\text{ time}}$

$mF^2 \longrightarrow mF \longrightarrow m^{3/2}$ 1973 2000s $\Rightarrow \tilde{O}(m^{3/2}/\varepsilon^2)$ time to find $(1+\varepsilon)$ apx max flow in undirected gr.

$m^{4/3}$ time 2010 flow algo

$\tilde{O}(m)$ algo 2012 (Gradient Descent based ideas)

"Truncated elec flow" / Elect flow with "fuses"