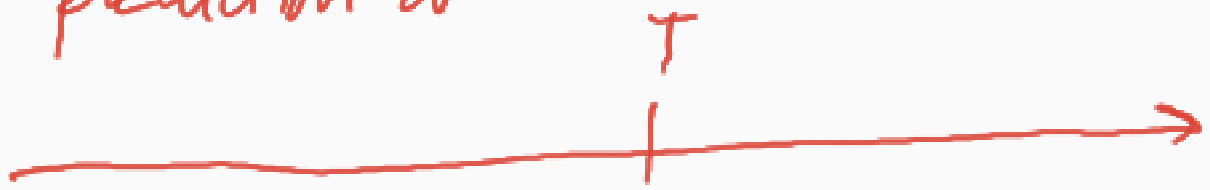
Lec 15: Experts to solve LPs

- Recall: experts problem of prediction
  - "regret" vs best expert in hindsight, Hedge algo.

N experts

prediction at t



- 0-sum games
- Linear Programs

solve apx.

Hedge Algo: $w^1 := \frac{1}{N}\mathbb{1}$

at round t

play $p^t = \frac{w^t}{\|w^t\|_1}$

observe ℓ^t

loss $= \langle \ell^t, p^t \rangle$

set $w_i^{t+1} \leftarrow w_i^t \, e^{-\varepsilon \ell_i^t}$

$[-1,1]^N$ Δ_N

$\langle \ell^t, p^t \rangle$ = loss@t

Theorem: for any sequence, $\ell^1 \ell^2 \dots \ell^T \in [-1, 1]^N$ $\quad \varepsilon \le 1/2$

$$\underbrace{\frac{1}{T}\sum_{t=1}^{T} \langle \ell^t, p^t \rangle}_{} \le \underbrace{\min_i \frac{1}{T}\sum_t \langle \ell^t, e_i \rangle}_{} + \underbrace{\varepsilon + \frac{\log N}{\varepsilon T}}_{}$$

$\le \varepsilon$

if T is $\ge \frac{\log N}{\varepsilon^2}$

$\Rightarrow$ sps $T \ge \frac{\log N}{\varepsilon^2} \Rightarrow \frac{\log N}{\varepsilon T} \le \varepsilon$

$\Rightarrow$ My avg loss $\le \min_i$ (Expert i's average loss) $+ 2\varepsilon$ if $T \ge \frac{\log N}{\varepsilon^2}$

$$\boxed{\text{losses} \in [-\rho, \rho]^N \Rightarrow \text{scale down} \Rightarrow \text{same guarantee but if } T \ge \frac{\rho^2 \log N}{\varepsilon^2}}$$

0 sum games: (2 player)

	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

n actions for C

m actions for R

M

$[-1, 1]$

strategy $p \in \Delta_m$ $\quad q \in \Delta_n$

payoff matrix

M_{ij} = amt of money C gives to R if C plays j, R plays i

$$E[\text{payoff}] = \underbrace{\sum_{ij} p_i q_j M_{ij}}_{= p^T M q}$$

← row player wants to max

col player wants to min

Sps R fixes a strategy p^*.

$$\min_{q \in \Delta_n} v^T q = \boxed{\sum_i v_i q_i}$$

col player ← $\min_{q} \boxed{p^{*T} M} q$

$$\boxed{C(p^*) := \min_j (p^*)^T M e_j}$$

↑ best resp

Sps C fixes q^*

$$\boxed{R(q^*) := \max_i (e_i)^T M q^*}$$

Fact: for any p, q $\boxed{C(p)} \leq R(q)$

↑ p

Thm: [minimax thm]

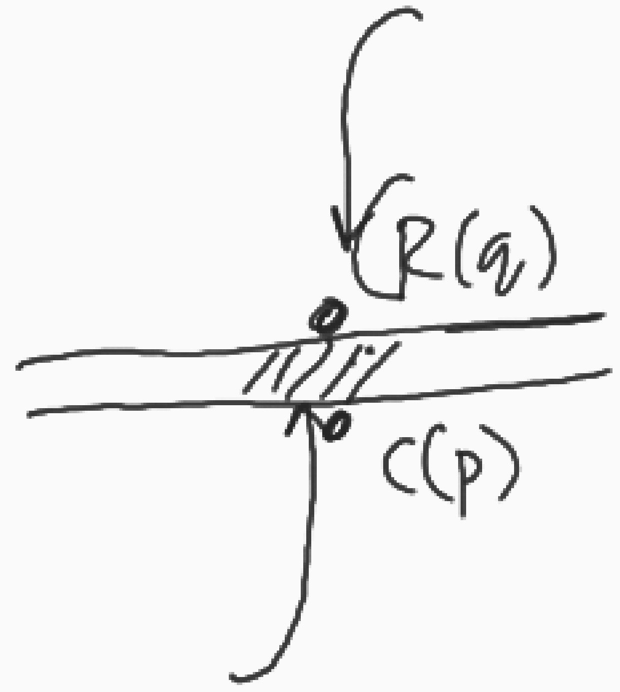
$$\max_p C(p) = \min_q R(q)$$

for any finite game

$$\max_p \min_q p^T M q = \min_q \max_p p^T M q$$

Pf: Sps not equal $\max_p C(p) < \min_q R(q) - \delta$ $\quad \delta > 0$.

$\exists M \in [-1,1]^{m\times n}$



at each time $t = 1 \ldots T$ $\quad p^1 \leftarrow$ unif

→ row player p^t on rows

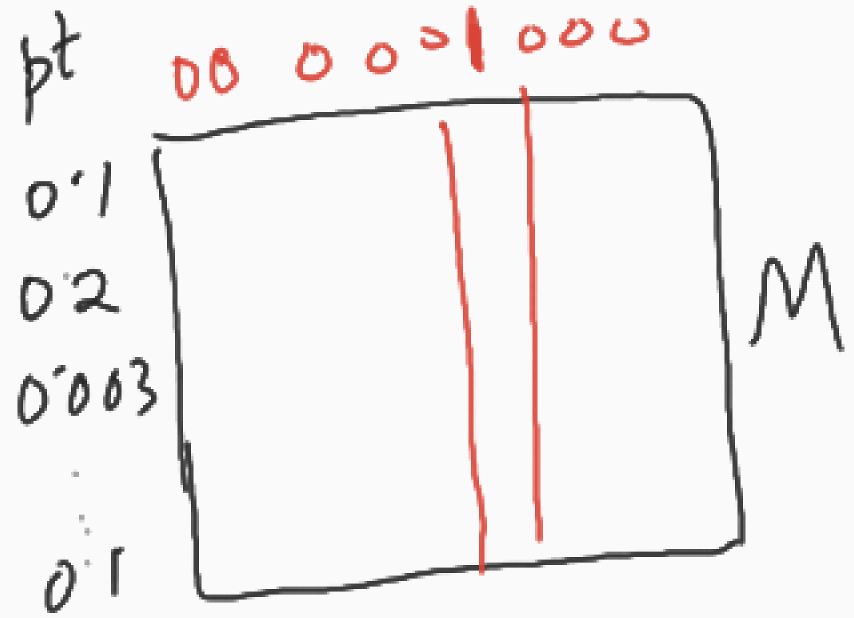
col player plays best response $q^t \leftarrow \underbrace{\arg\min_q (p^t)^T M q}_{e_{j^t}}$

use $Me_j = j^{th}$ col of M as $\cancel{\ell^t}$ g^t

feed into Hedge

↓

p^{t+1}



. $T = \frac{\log m}{\varepsilon^2}$

. $\hat{p} = \frac{\sum_t p^t}{T}$ $\qquad \hat{q} = \frac{\sum_t q^t}{T}$

$\forall p, q \quad C(p) \le R(q)$

Claim: $C(\hat{p}) \ge R(\hat{q}) - 2\varepsilon$

$\boxed{C(\hat{p}) \geq R(\hat{q}) - 2\varepsilon}$

$$\underbrace{\frac{1}{T}\sum \langle g^t, p^t \rangle}_{} \geq \max_i \underbrace{\frac{1}{T}\sum \langle e_i, g^t \rangle}_{} - 2\varepsilon \qquad \leftarrow \text{Hedge}$$

$$= \frac{1}{T}\sum C(p^t) \qquad\qquad = \frac{1}{T}\sum \langle e_i, M e_{j_t} \rangle$$

C is

Ex: concave

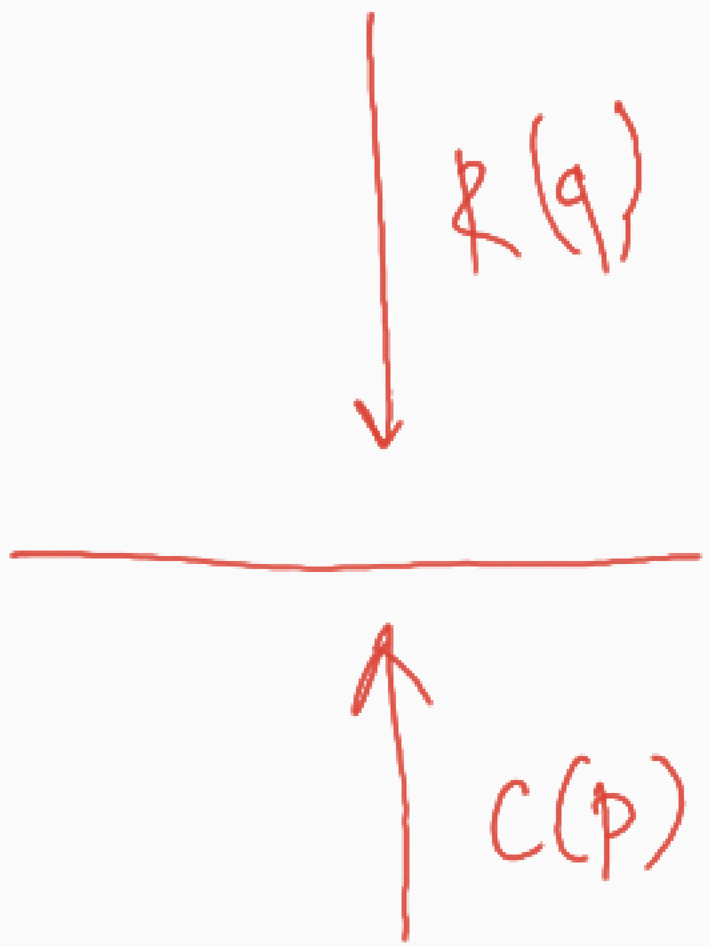
$$\leq C\left(\frac{1}{T}\sum p^t\right) \qquad\qquad = \langle e_i, \frac{M}{T}\underbrace{\sum g_t}_{} \rangle$$

$$= C(\hat{p}) \qquad\qquad = \underbrace{\langle e_i, M\hat{q} \rangle}_{}$$

$$\max_i \langle e_i, M\hat{q} \rangle$$

$$= R(\hat{q})$$

$$\Rightarrow C(\hat{p}) \geq R(\hat{q}) - \underline{\underline{2\varepsilon}} \qquad \blacksquare$$



LPs:

$$\left[\begin{array}{c} \langle c,x\rangle = OPT \leftarrow \\ \left[\,Ax \le b + \varepsilon \mathbb{1}\,\right]_{m \text{ constraint}} \\ x\ge 0 \end{array}\right] \text{ in } O\!\left(\frac{\log m}{\varepsilon^2}\right) \text{ "iteration"}$$

$c \cdot x$

$c^T x$

$\langle c, x\rangle$

✓ **Assume**: can solve 1 constraint

$$\boxed{\begin{array}{c}\langle c,x\rangle = OPT\\ x\ge 0\end{array}} \quad K$$

$$\boxed{\text{find } x \in \{\langle \alpha, x\rangle \le \beta \,,\ x\in K\}} \quad \text{"oracle"}$$

↖ 1 constraint problem easy.

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} \;A\; \end{bmatrix} \end{array} \begin{pmatrix} x \end{pmatrix} \le \begin{bmatrix} b \end{bmatrix}$$

to solve m constraints

$p^1 \in \Delta_m \qquad p^1 = (1/m, 1/m \;\dots\; 1/m)$

for $t = 1 \dots T$ do:

average all constraints acc to p^t

$$\sum_i p_i^t \langle a_i, x \rangle \le \sum_i b_i p_i^t \longrightarrow \langle \alpha^t, x \rangle \le \beta^t$$

find $x^t \in \{ \langle \alpha^t, x \rangle \le \beta^t, \; x \ge 0, \; \langle c, x \rangle = OPT \}$

$$g_i^t = \langle a_i, x^t \rangle - b_i$$

Feed to hedge

$$\begin{array}{cc} & n \\ \begin{array}{c} 1/10 \to \\ 1/10 \to \\ m \\ 1/10 \to \end{array} & \begin{bmatrix} \text{---} \; a_1 \; \text{---} \\ \text{---} \; a_2 \; \text{---} \\ A \\ \text{---} \; a_m \; \text{---} \end{bmatrix} \end{array} \begin{pmatrix} x \end{pmatrix} \le \begin{bmatrix} b \end{bmatrix}$$

width

$$T = \frac{\rho^2 \cdot \lg m}{\varepsilon^2}$$

poly$(1/\varepsilon)$ vs poly $\lg\left(\frac{1}{\varepsilon}\right)$

where ρ is an upper bound on $|\langle a_i, x^t \rangle - b_i|$

Thm: $\hat{x} = \frac{x^1 + x^2 + \dots x^T}{T}$ sats $\langle c, \hat{x} \rangle = OPT$, $\hat{x} \ge 0$

$A\hat{x} \le b + 2\varepsilon \mathbb{1}$