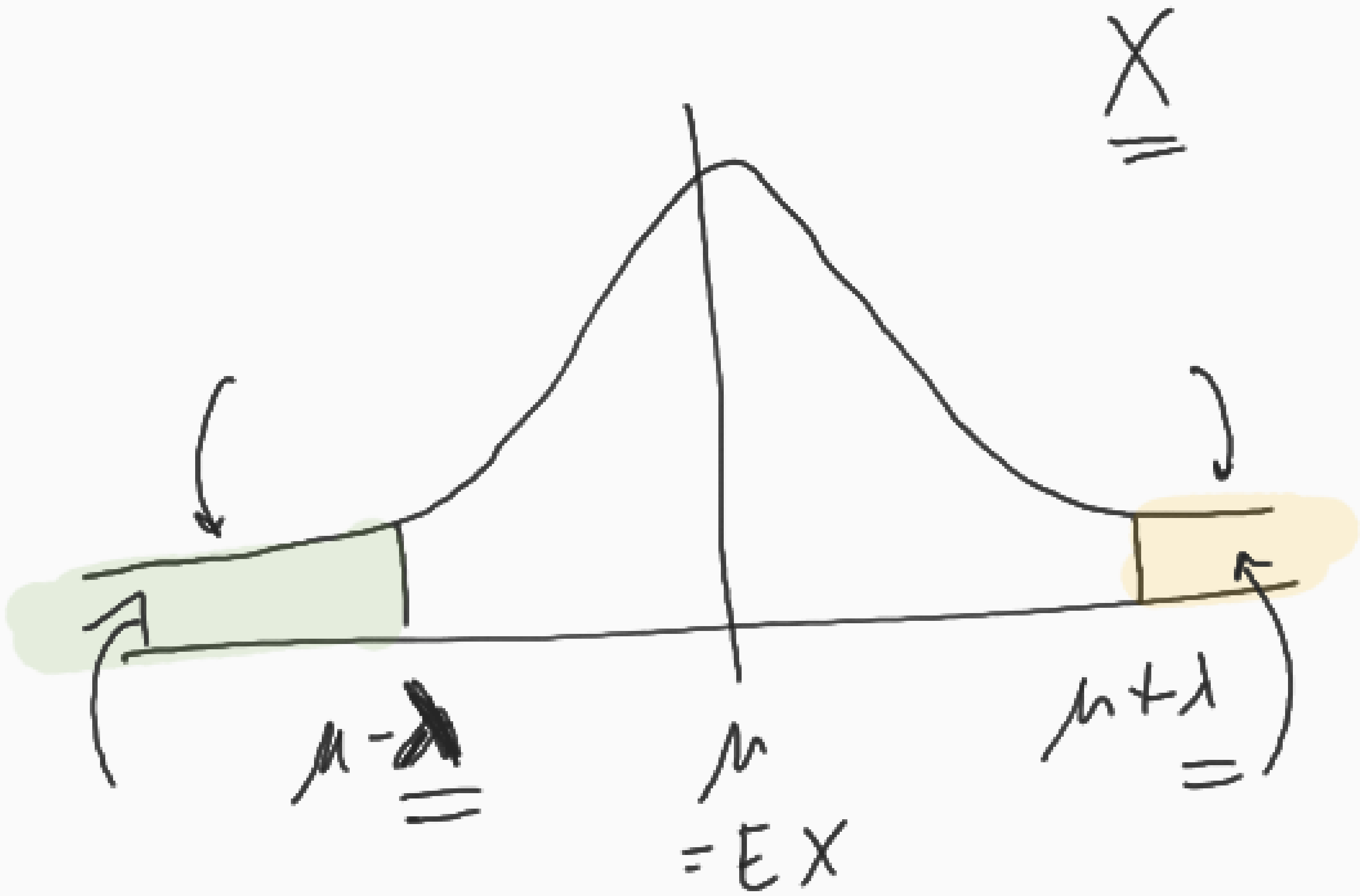
Lecture #11 : Concentration Bounds II

- Hypercube Routing
- Dimension Reduction

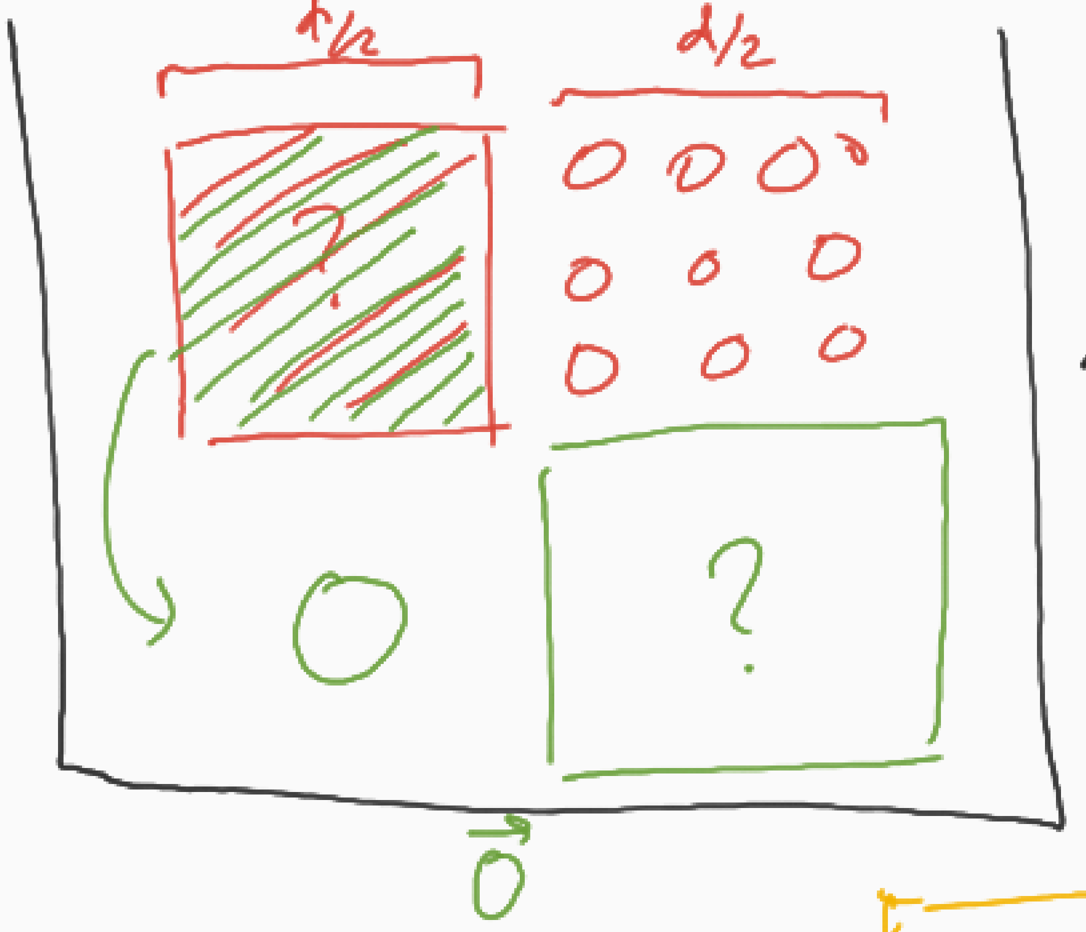


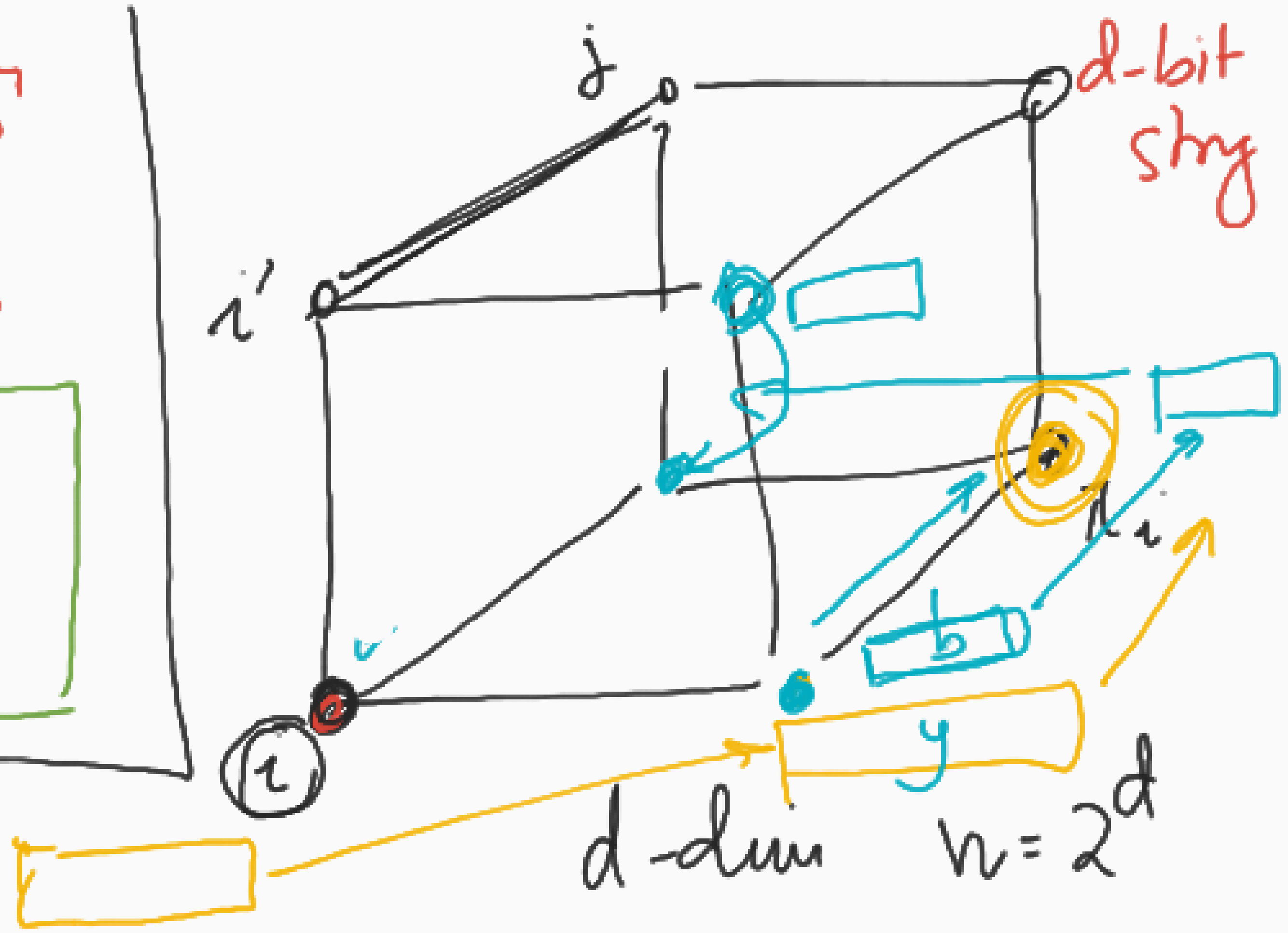
$X_1, X_2, \ldots \in [0,1]$ independent

$S = \sum_i X_i$ with $\mathbb{E}[S] = \mu$

then $\Pr[S \geq \underbrace{\mu}_{\text{mean}} + \underbrace{\lambda}_{\text{deviation}}] \leq \underbrace{\exp\left(-\frac{\lambda^2}{2\mu+\lambda}\right)}_{\text{small! (hopefully)}}$

Constraint



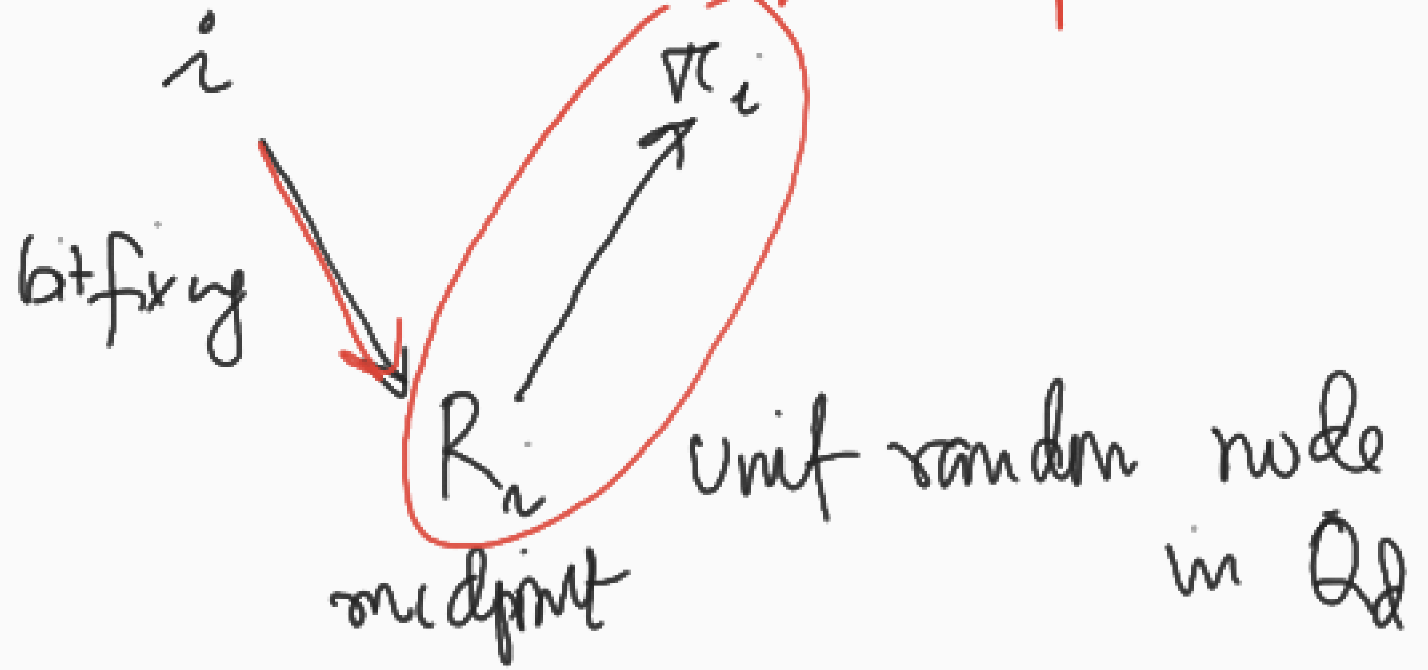


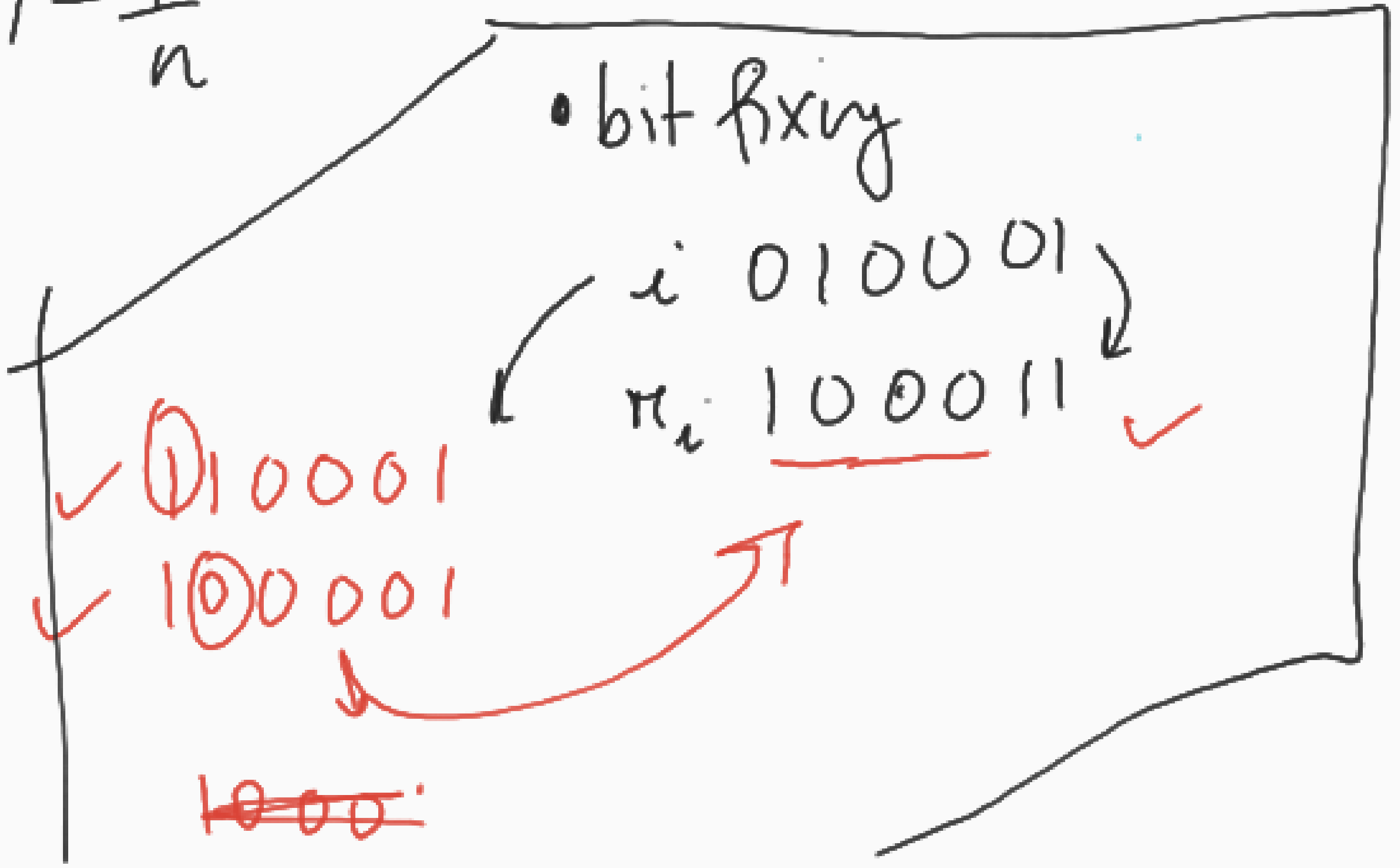
Goal: $O(d)$ steps with high prob $\left(1-\frac{1}{2^d}\right) = 1-\frac{1}{n}$

π permutation

Thm all packets get to midpts $\le 5d$ steps whp.

Algo





$i \longrightarrow R_i$

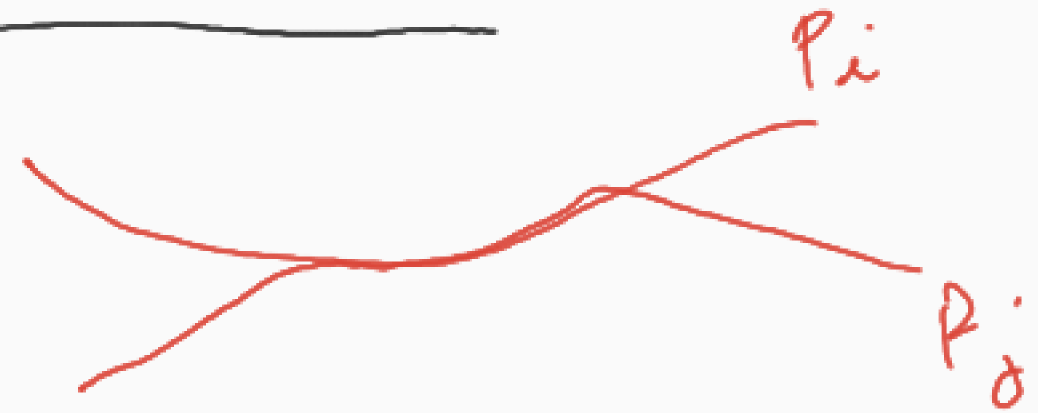
P_i : bit fixing path $i \longrightarrow R_i$

$E[\text{length of } P_i] = E[\#\text{ of locations that } i \,\&\, R_i \text{ differ}] = d/2$.

$\#\text{ edges} = 2^d d$ $\qquad E[\#\text{ total length of } 2^d \text{ paths}] = 2^d \cdot d/2$

$\Rightarrow$ Symm, each edge has $E[\#\text{ paths}] = \dfrac{(2^d \cdot d/2)}{2^d d} = \underline{1/2}$.

$\boxed{\text{Fact:}}$ P_i & P_j



meet in contiguous segment

$S(i) = \{$ paths P_j that ~~intersect~~ share an edge with $P_i \}$

__Lemma 1__: packet i reach its destination @ time $\leq |P_i| + |S(i)|$

$\boxed{\underline{\text{Lemma 2}}.\ |S_i| \leq 4d \text{ with high prob.}}$

Lem 2. $\Pr(|S(i)| \geq 4d) \leq e^{-2d}$

Pf: $|S(i)|$ $\qquad \mu = E[|S(i)|] \leq d/2$

$$S(i) = \underbrace{\sum_{j \neq i} \mathbb{1}(P_i \,\&\, P_j \text{ intersect in an edge})}_{\text{sum of indep. bounded r.v.s.}}$$

$\{0,1\}$ rvs

$\Pr[S \geq \mu + \lambda] \leq$ ______

$\Pr[S \geq \mu + (4d - \mu)]$

$\leq \exp\left(-\frac{(4d-\mu)^2}{2\mu + (4d-\mu)}\right)$

$\leq \exp(-2d)$

$\Rightarrow \Pr[\exists i \text{ st } |S(i)| \geq 4d] \leq 2^d \cdot e^{-2d} \leq e^{-d}$

$\leq 2^{-d}$

Lemma 1: packet i reaches R_i by time $\leq \boxed{|P_i| + |S(i)|}$

$L \to L+1$

$t - k$

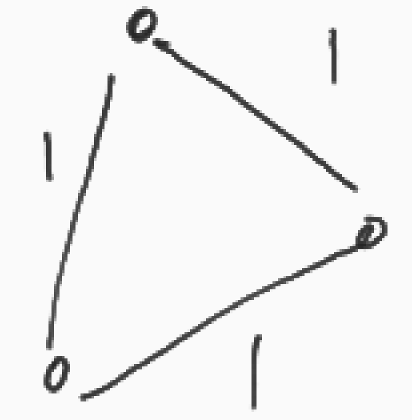
Oblivious

Valiant & Brebner

Dimension Reduction & JL

$\varepsilon \in [0, 1/2)$

Claim: n pts in $\mathbb{R}^D$ $\quad x_1, x_2 \dots x_n \quad$ $\|x_i - x_j\|_2 = 1$



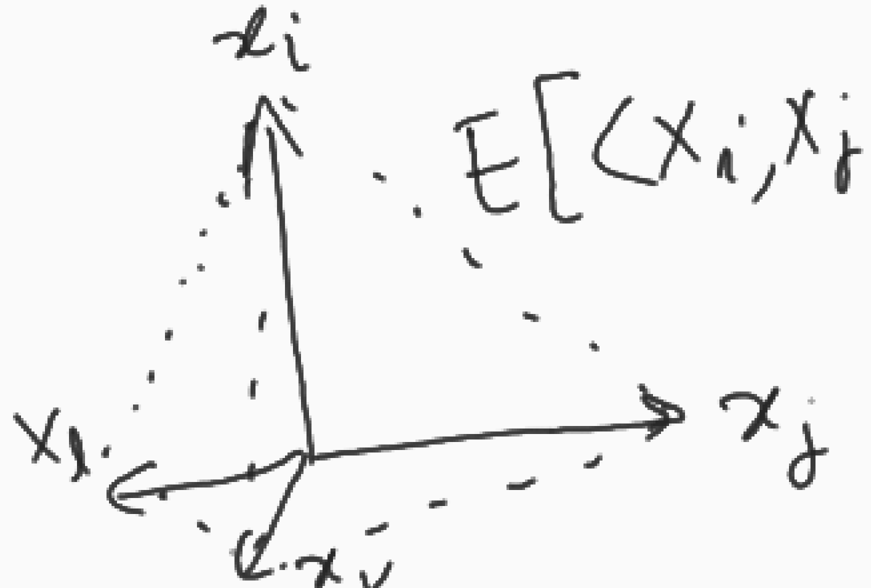
A - 1

n pts $\quad \|x_i - x_j\|_2 \in [1-\varepsilon, 1+\varepsilon] \quad \forall ij$

smallest dim D

achieve in $D \simeq O\left(\frac{\log n}{\varepsilon^2}\right)$

$\Leftrightarrow$ # of almost (±ε) equidistant pts in $\mathbb{R}^D$ is $\boxed{\exp(c \cdot \varepsilon^2 \cdot D)}$



$E[\langle x_i, x_j \rangle] = E\left[\sum_{k=1}^{D} x_{ik} x_{jk}\right] = 0$

$\boxed{\text{n random } \{\pm 1, 1\}^D \text{ vectors unif from}}$

n pts from ~~D~~ $\{\pm 1\}^D$ $\qquad n = 2^{c \cdot \varepsilon^2 D}$

then $\Pr\left[\forall i,j \quad \frac{\|x_i - x_j\|}{\sqrt{2D}} \in (1 \pm \varepsilon)\right] \geq 1/2$

Johnson Lindenstrauss Flattening Theorem (1984)

Given $x_1, x_2 \ldots x_n \in \mathbb{R}^D$, there exists a linear map $\varphi : \mathbb{R}^D \to \mathbb{R}^k$

s.t. $1 - \varepsilon \leq \frac{\|\varphi(x_i) - \varphi(x_j)\|_2}{\|x_i - x_j\|_2} \leq 1 + \varepsilon \qquad \forall i,j$

10^{10} n

2^{10} D

$\frac{10 \lg 10}{\varepsilon^2}$

and $k = \frac{O(\log n)}{\varepsilon^2}$.

$\varphi(x+y) = \varphi(x) + \varphi(y)$

And $\exists$ algo to find this map φ.

$x_1 \ldots x_n \in \mathbb{R}^D \to \mathbb{R}^K$

M is a Gaussian matrix

$$\varphi(x) = \frac{1}{\sqrt{K}} \underbrace{K\left\{\begin{bmatrix} \square & \square \end{bmatrix}\right.}_{D}^{M} \begin{pmatrix} x \end{pmatrix} = \boxed{\frac{1}{\sqrt{K}} Mx}$$

$\sim N(0,1)$ Gaussian

Claim: φ satisfies thm whp.

$K = \Theta\left(\frac{\log n}{\varepsilon^2}\right)$

$$\boxed{\text{One Vector Version: sps. } x \text{ is unit vector then } \|\varphi(x)\|_2 \in [1 \pm \varepsilon] \text{ w.p. } 1 - \frac{1}{n^2}}$$

$\Rightarrow$ JL $\quad \dfrac{\|\varphi(x_i) - \varphi(x_j)\|}{\|x_i - x_j\|} = \dfrac{\|\varphi(x_i - x_j)\|}{\|x_i - x_j\|} = \left\|\varphi\left(\dfrac{x_i - x_j}{\|x_i - x_j\|}\right)\right\| \in [1 \pm \varepsilon]$ w.p. $1 - \frac{1}{n^2}$

union bound $\Rightarrow$ all $\binom{n}{2}$ distances maintained w.p. $1 - \binom{n}{2} \cdot \frac{1}{n^2} \geq \frac{1}{2}$

x unitvector

$\|\varphi(x)\| \in [1 \pm \varepsilon]$ whp.

$\varphi(x) =$

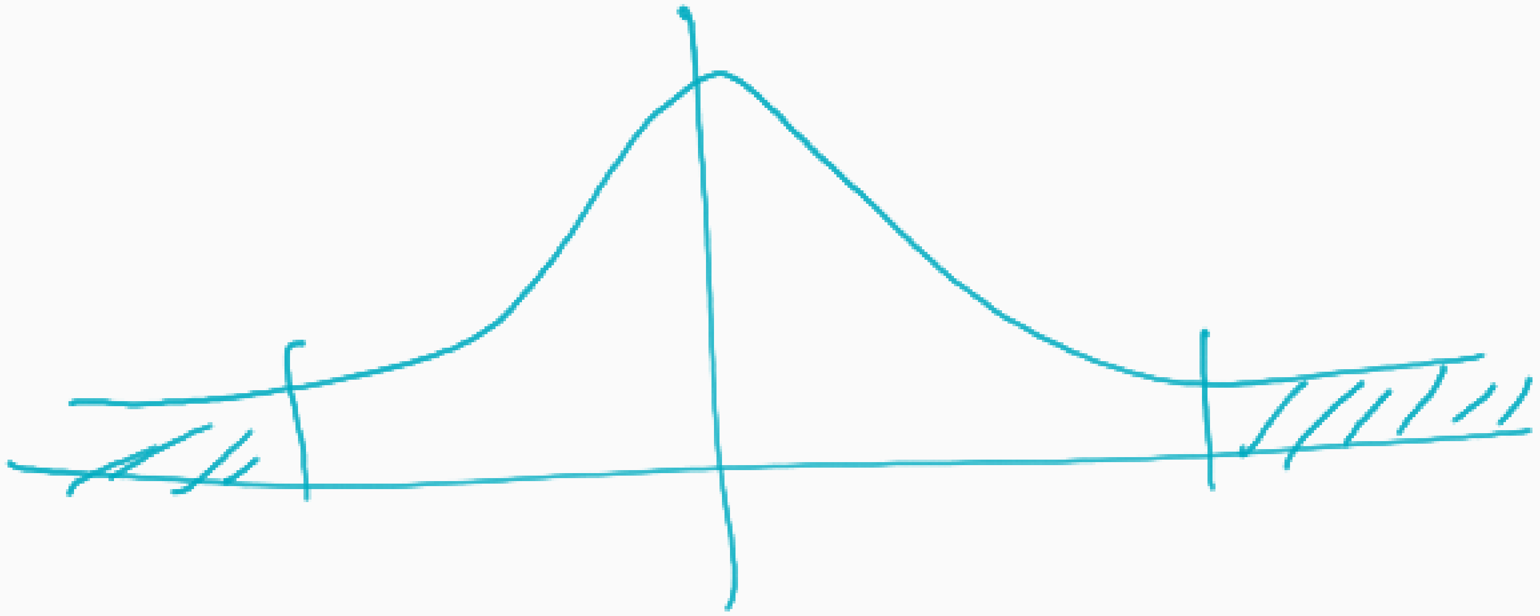
(1) $\boxed{x = e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}$

$$\varphi(x) = \frac{1}{\sqrt{K}} \begin{pmatrix} & M & \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{K}} \cdot \begin{pmatrix} M_1 \end{pmatrix} = \frac{1}{\sqrt{K}} \underline{\underline{(G_1 \; G_2 \; \ldots \; G_k)^T}}$$

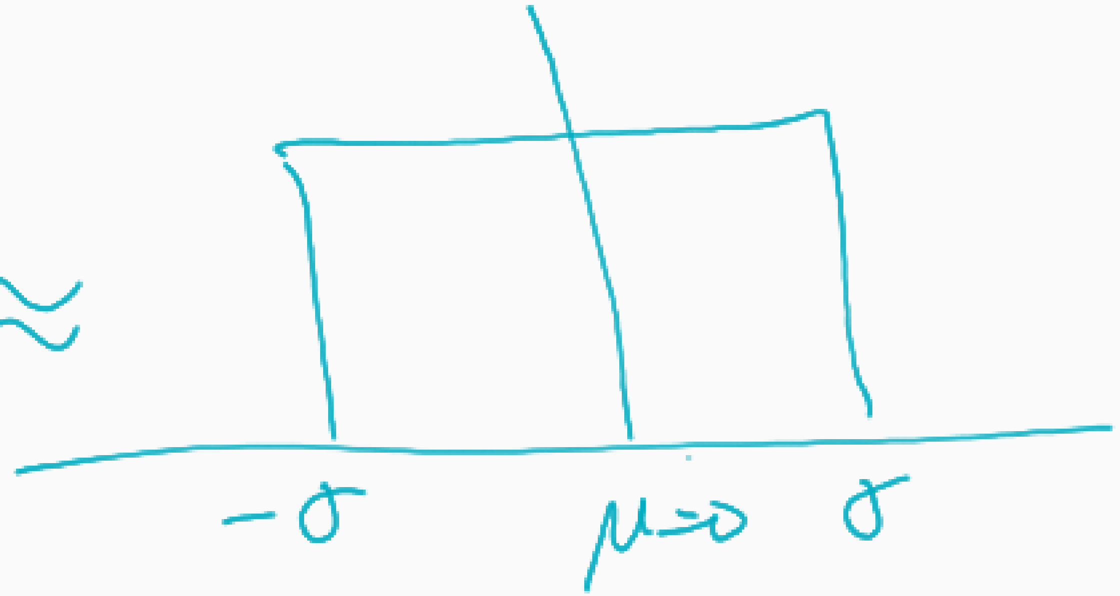
$$\|\varphi(x)\|_2^2 = \frac{1}{K} \sum_{i=1}^{K} G_i^2 = \text{Average of a sum of (Gaussian squared)}$$

bounded?

$=$

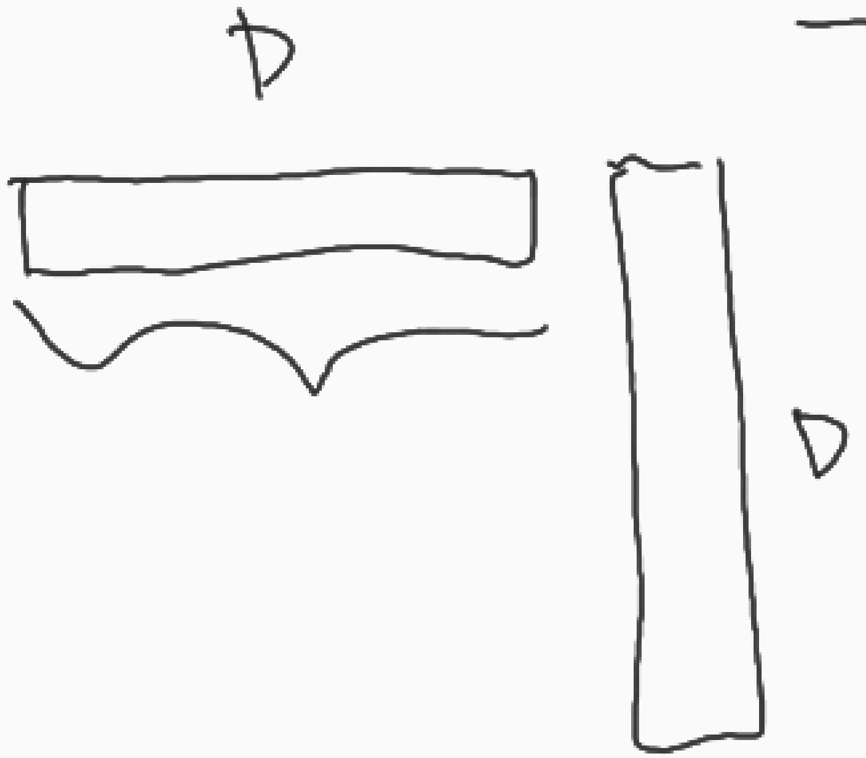


$\approx$



$D \times D$

$A^T B$



$$\underline{(MA)^T(MB)} = A^T \underbrace{M^T M}_{\frac{\log D}{\varepsilon^2}} B \cong A^T B$$