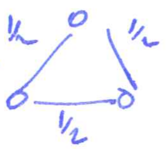
What about non-bipartite graphs?

Also: $\left\{ \sum_j x_{ij} = 1 \quad \forall i \in V, \quad x_{ij} \geq 0 \right\}$ is not the same as convexhull of all perfect matchings.

Eg.  is a solution to the polytope, but $\nexists$ no perfect matching in C_3.

Edmonds: the perfect matching polytope (i.e. convex hull of all PMs in G) is given by

$$K_{PM} = \left\{ x \in \mathbb{R}^{E} \;\middle|\; \begin{array}{ll} x(\partial v) = 1 & \forall v \\ x(\partial S) \geq 1 & \forall \text{ odd sets } S. \\ x \geq 0 & \end{array} \right\}.$$

Proof: Different ways to do it. Can use Blossom algorithm. Here is direct proof.

Let CH_{PM} = convex hull of all χ_M : M = perfect matching in G.

Since each $\chi_M \in K_{PM}$, $CH_{PM} \subseteq K_{PM}$. So now suffices to show that $K_{PM} \subseteq CH_{PM}$. We induct on $|E|$.

Base Case: $|E| = 1$ then must have 2 vertices uv, and $x_{uv} = 1$. $\Rightarrow$ trivial

Inductive Step: $\bar{x} \in K_{PM}$. and $\bar{x}$ is a vertex of K_{PM}.
want to show $\bar{x} \in CH_{PM}$

then if all vertices of K_{PM} in CH $\Rightarrow$ all of $K \subseteq CH$.

if $\exists e$ st. $x_e = 0$ then induct on $G \backslash e$.
$x_e = 1$ then induct on $G \backslash \{u, v\}$ since all other edges in $\partial u, \partial v = 0$.

$\forall e \in E$ $0 < x_e < 1$ then if all vertices have degree 2 $\Rightarrow$ x cannot be a vertex of K_{PM}.
$\Rightarrow$ cycles

$\Rightarrow \exists$ vertex of degree ≥ 3 (and all else ≥ 2). $\Rightarrow |E| > |V|$. $\Rightarrow$ $\geq n+1$ tight constraints.

$\Rightarrow \exists$ one non-trivial constraint tight S^* odd $x(\partial S^*) = 1$.

$\bar{S} = V \setminus S^*$. G/S^*, $G/\bar{S}$ by contracting one side or other to vertex. (7)

x^*, $\bar{x}$.

Since $x(\partial U) = 1$ both are in the K_{PM} polytope of the respective graphs.

$$\Rightarrow x^* = \sum_{M \text{ pm in } G/S^*} \alpha_M \chi_M \qquad \text{make } \tfrac{1}{N} \Rightarrow = \frac{1}{N} \sum \chi_M$$

$$\bar{x} = \sum_{N \text{ pm in } G/\bar{S}} \beta_N \chi_N . \qquad = \frac{1}{N} \sum \chi_N$$

Now match them up to get $x = \frac{1}{N} \sum_{\tilde{M} \in PM \text{ of } G} \chi_{\tilde{M}}$.

■

Here's a different proof that for bipartite graphs, the perfect matching polytope is

$$K = \{ x \in \mathbb{R}^E \mid \sum_i x_{ij} = 1 \quad \sum_j x_{ij} = 1 \} \subseteq \mathbb{R}^m$$
$$x_{ij} \geq 0$$

Pf: ~~CH be the convex hull~~

Consider any vertex of K. Want to show it is a perfect matching.

(Since $\forall M$, $\chi_M \in K$, this will prove that $K = CH(PMs)$).

$x \in K$ is a vertex. So obtained by m tight constraints (linearly indep.)

~~Some~~ there are $2n + m$ constraints.

Also $\leq (2n-1) + m$ LI constraints (since $\sum_j \left(\sum_i x_{ij} = 1 \right) = \sum_i \left(\sum_j x_{ij} = 1 \right)$)

$\Rightarrow$ at ~~least~~ most $2n-1$ of the interesting constraints are tight at x (and all must be)

~~rest~~ at least $m - (2n-1)$ of the tight constraints at x are $x_{ij} = 0$.

$\Rightarrow$ at most $2n-1$ edges have non zero values.

But $\exists$ $2n$ vertices. and each vertex has ≥ 1 edge out of it

~~So this a of~~ there is a vertex with degree 1.

$\Rightarrow \exists$ edge with value $= 1$. $\Rightarrow$ ~~$\exists$ another vertex~~

$x_{uv} = 1$

Now induct on the rest of the graph $G \setminus \{u, v\}$.

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