Lecture 16: Online Decision Making II

Difficulties not (just) computational
but lack of information

Decision Making in the face of
Uncertainty

\[ \text{find } \text{Alg} \text{ s.t. } \left( \max_{I} \frac{\text{cost Alg}(I)}{\text{OPT}(I)} \right) \leq \text{small competitive ratio} \]
Mistake Baylor Model | Experts Problem

$\forall \text{day } t = 1, 2, \ldots, T$

- So listen to n "experts" prediction
- make a prediction $\alpha_t$
- afterwards, hear the actual outcome $O_t$

$\text{Mistake}_t = \alpha_t \neq O_t$

Minimize # mistakes
1. Sps a perfect expert.
   \[ \Rightarrow \exists \text{ algo that makes } \leq \log_2 n \text{ mistakes} \]
   Predict the majority opinion, discard all incorrect experts.
   Every mistake \( \Rightarrow \) throw away majority of earliest expert
   \[ \Rightarrow \leq \log_2 n \text{ mistakes} \]

2. Sps the best expert makes \( m^* \) mistakes
   \[ \Rightarrow \exists \text{ algo that makes } \leq (m^* + 1)(\log_2 n) \text{ mistakes}. \]
   \[ m^* = \min m_i \]
   \[ m_i \leq (\#\text{rnds}) \log_2 n + 1 \]
   \[ m^* \geq (\#\text{rnds} - 1) \text{ mistakes} \]
For expert $i$, $Ago's$ metades $\leq \frac{\text{mut}}{\frac{\log_2 n}{2}} \left( \text{Expert } i \text{'s metades} \right) + \left( \frac{\log_2 n}{\text{necc.}} \right)$. 

Then for expert $i$, $Ago's$ metades $\leq \frac{1}{t} \left( 1+\varepsilon \right) \left( \text{Expert } i \text{'s metades} \right) + \left( \frac{\log_2 n}{\varepsilon t} \right)$. 
\[ w^{(i)}_t = 1 \quad \text{if expert } i \]

for \( t = 1, 2, 3, \ldots, T \ldots \)

1. Predict the weighted majority
   
   - \( \text{if } \sum_{i} w_t^{(i)} \geq \sum_{i} w_t^{(i)} \text{ predict } (y) \)
   - \( \text{else } \neg y \)
   
   \[ w_{t+1} = \begin{cases} 
   w_t^{(i)} & \text{if } (y) \text{ correct} \\
   w_t^{(i)} \cdot \frac{1}{2} & \text{if } \neg y 
   \end{cases} \]

   **Weighted Majority (WM)**

   **Multiplicative Weight (MW)**

**Theorem:** Mistakes of WM \( \leq 2.4 \) (Mistakes of i) + \( 0(\log n) \)

**Proof:** "potential" \( \Phi_t = \sum_i w_t^{(i)} \quad \Phi_0 = n \)

- If \( \Phi_t \) make no mistake in \( t \)
  - \( \Phi_t^{\text{mist}} \leq \Phi_t \)
- If \( \Phi_t \) make mistakes \( \Phi_t^{\text{mist}} \leq \frac{3}{4} \Phi_t \)

\[ \left( \frac{1}{2} \right)^{\text{mist}} \leq \Phi_t \leq \left( \frac{3}{4} \right)^{\text{mist}} \quad \Phi_0 \leq n \cdot \left( \frac{3}{4} \right)^{\text{mist}} \]

\[ \left( \frac{1}{2} \right)^{\text{mist}} \leq n \cdot \left( \frac{3}{4} \right)^{\text{mist}} \Rightarrow \left( \frac{4}{3} \right)^{\text{mist}} \leq n \cdot 2^{\text{mist}} \]

\[ \text{take logs } \Rightarrow A^{\text{mist}} \left( \log_2 \frac{1}{3} \right) \leq \log n + \text{mist} \]
1. Basic MW/MW with halving gain. 
   \[ A_g \leq \frac{2^i}{m_i} + O(\log n) \]

2. \[ W^t_i \leftarrow W^t_i (1-\varepsilon) \]

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Total loss of three experts \( \leq 1/2 \) mistakes.

**Bad news:** No deterministic algo can do better than 2 \( m_i \).

**PF 2 experts**

| Away | YES \| Ayes | NO |
|------|--------|------|----|

outcome \( t \neq \alpha t \) => Also makes \( T \) mistakes.

So, the total loss of three experts \( \leq 1/2 \) mistakes.
Shown: $\text{LMAE} \leq \text{HE MWM}$

$\bullet$ $z$ is made

$\bullet$ $z$ is used

$\bullet$ $z$ is revealed

$\bullet$ $z$ is as predicted

Model (Refined)

$\sum_{i=1}^{n} w_i 
= \sum_{i=1}^{n} (1 - E) \cdot p \cdot z$

$w_i = \delta w_i \cdot (1 - E)$

Rand MWM: [Littlestone/Mannila]
Extension 1: Change of Perspective

at each timestep \( t_i \),

- Algorithm produces a vector \( p^t \in \Delta_n \)

- Nature produces a vector \( l^t \in \{0,1\}^n \)

My cost at time \( t \) = \( \langle l^t, p^t \rangle \)  
\[
\sum_{i} l_i^t p_i^t
\]

Wait! My loss over time
\[
\sum_{t} \langle l^t, p^t \rangle
\]

\( \leq \) loss of best vector \( q^* \in \Delta_n \)

\[
= \min_{q^* \in \Delta_n} \sum_{t} \langle l^t, q^* \rangle + \frac{\log n}{\varepsilon}
\]
Ex2: Check that linear function model (dot product model) captures experts model.

"Bandits" instead of "experts" / partial info model.

- In previous models, see online loss vector.
- Now suppose I only see the loss value $\langle l_t, p_t \rangle$
- Or in RWM model, pick a random expert, get see its loss.

Good news: Experts algs extend to bandits.

$$H\delta^* = E[\text{loss of alg in bandit setting}] \leq (H\delta) \sum_t \langle l_t, q^* \rangle + \sqrt{\frac{O(1/\epsilon)}{t}}$$

→ explore vs exploit