- The network flow model
- Application
  - Algorithms
- Airplane Scheduling
  - Baseball Elimination
  - Project Scheduling
Network Flows

$G$ is a flow network directed.

- $u_e$ capacity
- "upper bound"

$\text{flow: } f : E \rightarrow \mathbb{R}$

$s.t.
\begin{align*}
1 & \quad 0 \leq f(e) \leq u_e \quad \forall e \\
2 & \quad \forall \text{ node } v \neq s, t \quad (\text{conservation of flow})\\
& \quad \sum_{u \in \text{pre-}v} f(uv) = \sum_{w \in \text{post-}v} f(vw)\\
3 & \quad \text{no flow enters } s, \text{ no flow leaves } t \\
\end{align*}$
Theorems: Given any flow network, a max value flow (maxflow) can be found in poly time.

Ford-Fulkerson (not poly time) $\rightarrow$ good for simple cases

[Edmonds-Karp, Dinics, Goldberg-Rao] $\rightarrow$ poly time

Thm 2: If all capacities are integers, then there is a max flow that sends integral integer flow on all edges. $f: E \rightarrow \mathbb{Z}^+$ (and found in poly time)

Algs for Max Flow

Implicit max flow.
Flow.

Flow Decomposition Theorem:
Every flow can be decomposed into $m$ flows along paths.

Moreover, if the flow values are integers, then each path sends integer-valued flow.
Application #1: Jobs & Machines (Bip Matching)

- Each job $j \rightarrow$ 1 machine
- Each machine takes $\leq 1$ job

Also:
1. Find a max integer flow in $G$
2. Paths down
3. Return edges used in this flow

Max Flow $F^*$

\[ \text{Max Match.} \]

\[ \cup \quad \cap \quad \lor \quad \land \quad \forall \]

Example where greedy fails (to come).
Airline Schedule:

I. Multiple source cities.

\[ \sum b_i = 0 \]
(c) Lower bounds on edge flows.

The upper bound / capacity \( U_e \) is

\[ l_e \leq f(e) \leq U_e \]

\[ b_i^0 \rightarrow a \]

\[ b_j \rightarrow b_i + 3 \]

redue to multiple source sink case
(d) Airplanes (12 of them)  | Initial locations, final locations

- 9AM  | 10:30  | 12  | 2PM
- PIT  | EWR   | PHL | JFK

Plane 1 @ PIT

Required

Start  | 21  | 6

PIT  | 0

EWR  | 0

PHL  | 0

JFK  | 0

-1  | 0  | -1  | -1
Max Flows

- can solve efficiently
- integer max flows (for integer capacities)
- model other extensions (multiple source sinks, lower bounds)

- max flow/min cut duality
- algorithms
Fact 1: A flow \( f \), any s-t cut \( C \)

\[ \text{value}(f) \leq \text{value}(C) \]

Fact 2: [MaxFlow/MinCut theorem]

For flow \( f^* \), cut \( C^* \)

\[ \text{value}(f^*) = \text{value}(C^*) \]