Useful Probability Exercise

Bucket has green (good) and red (bad) balls. You pick a
G balls / B balls. ($G, B > 0$)

uniformly random one. Win if green.

(Q1). Prob of victory?

$G/(G+B)$.

G good outcomes
B bad ones.
each has same probability

Ok. Now some white balls in there too. (W of them)

Process changes

If you get a white ball you throw it away (do not put back into bucket.) and try again

if green, you win. Red, you lose.

(Q2) Prob of victory?

Claim: Still $G/(G+B)$

Let's try small cases: $W=0$, same as above.

$W=1$: Let's see what happens on first draw.

Case: 1st ball is white. ⇒ throw away, and get back to Q1.
⇒ $G/(G+B)$

Case: 1st ball is not white

⇒ conditioned on that, have G good outcomes
B bad ones

⇒ $\Pr(\text{victory} \mid \text{1st ball not white}) = G/(G+B)$.

In both cases we win with prob $= G/G+B$

$\Rightarrow$ it happens overall.

General Case can be proven by induction in the same way.

Let $\mathcal{E}_1$ = event that we get a <u>non-white</u> ball in 1st draw.

$Pr[\text{victory} \mid \mathcal{E}_1] = G/G+B$ G good outcomes, B bad ones $\Rightarrow G/G+B$.

$Pr[\text{victory} \mid \neg \mathcal{E}_1] = G/G+B$ by induction since we throw away the ball and try again

$$Pr[\text{victory}] = Pr[\text{victory} \mid \mathcal{E}_1] \cdot Pr[\mathcal{E}_1] + Pr[\text{victory} \mid \neg\mathcal{E}_1]\, Pr[\neg \mathcal{E}_1]$$

$$= (G/G+B)\, Pr[\mathcal{E}_1] + (G/G+B)\, Pr[\neg \mathcal{E}_1]$$

$$= (G/G+B)\, [\underbrace{Pr[\mathcal{E}_1] + Pr[\neg \mathcal{E}_1]}_{=1 \text{ by Law of total Probability}}]$$

$$= \frac{G}{G+B}. \quad ☺.$$

<u>Exercises</u>: (Q3). Suppose each time you get white, you throw away some number of other whites as well. Show answer remains unchanged

(Q4) Suppose you don't throw away whites, you put them back. i.e. you sample <u>with</u> replacement. Show answer remains unchanged.