Rules: Work in groups of 3. Submit one set of solutions per group, and remember to put all your names on it. We'd prefer if you typed your answers, but if your handwriting is neat and legible, written work is fine. You may refer to whatever sources you want, but *CITE* your references! If you get a great idea from somebody, give appropriate credit: it won't lower your grade, and it's the right thing to do. (Note: Ryan *will* grade both of the questions.) If you have any questions, email `ryanw@cs.cmu.edu`.

1 Spanning trees for edge types.

Let G be a connected undirected graph, where each edge is colored either black or white. Suppose we wish to connect all the nodes together, and the cost of black edges "offset" the cost of white edges. Our goal will be to find a spanning tree T such that the number of black edges in T is as close as possible to the number of white edges in T.

More precisely, letting $b(T)$ and $w(T)$ be the number of black edges and white edges respectively in an arbitrary subgraph T, the problem is to find an spanning tree T of G such that $|b(T) - w(T)|$ is minimized.

1. Give a polynomial time algorithm for this problem.

2. Prove its correctness; i.e. prove your algorithm returns T^* with $|b(T) - w(T)| = k$ if and only if there exists a T such that $|b(T^*) - w(T^*)| = k$ and there is no T' such that $|b(T') - w(T')| < k$.

3. Give a sketch of the time complexity for your algorithm.

2 Cuts and cycles.

This problem is meant to show you how general the definition of matroid is, by working with a concrete example that has properties very different from the matroids on graphs we have looked at.

Let r and n be natural numbers such that $r < n$. Define $[n] = \{0, 1, \ldots, n\}$, and let $I = \{S \subseteq [n] : |S| \leq r\}$.

1. Prove that $([n], I)$ is a matroid.

2. What is this matroid's dual? Prove your claim.

3. Prove that the dual of the dual you obtained in 2 is actually $([n], I)$.

4. Suppose n is even (note this assumption is not required). Prove that if $r \leq n/2$, then for any $j = 2, \ldots, r + 1$, there exists a cut C and cycle D in $([n], I)$ such that $|C \cap D| = j$. That is, C and D intersect in exactly j elements.

3 Open problem of the week.

The open problem of the week is *NOT* something you're expected to solve or turn in for your homework. It is simply a research problem for you to think about when the lecture gets boring. If you solve one of them, you will have a nice conference paper/thesis.

This week's is: *Is there an algorithm for finding minimum spanning trees in $O(m+n)$ (i.e. linear) time?* There is such an algorithm (cf. Karger et al.) which uses randomness (i.e. it assumes access to an unbiased coin that can be flipped), but it is open if a purely *deterministic* algorithm exists. The best known deterministic algorithm is very close to this (cf. Chazelle). Perhaps some sophisticated uses of the Red and Blue rules could lead to a linear time algorithm– or perhaps it is impossible to find an MST in such a short time.

References:

David R. Karger, Philip N. Klein, Robert E. Tarjan, *A randomized linear-time algorithm to find minimum spanning trees.* Journal of the ACM (JACM), 42(2): 321-328, 1995.

Bernard Chazelle, *A minimum spanning tree algorithm with Inverse-Ackermann type complexity.* JACM 47(6): 1028-1047, 2000.