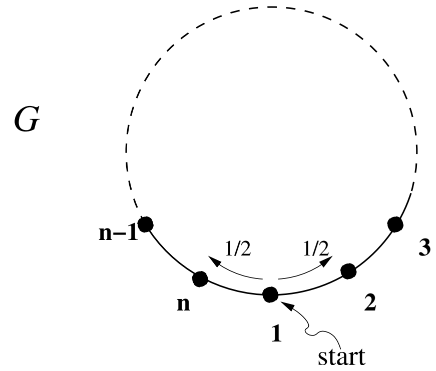
Problem 1: Random Walk on a Cycle (20 points)

(this problem is due to Avrim Blum)

Let G denote graph which is a cycle on n vertices (see Figure below). View G as a Markov Chain, i.e. consider a random walk on G starting at vertex #1, where with probability 1/2 you move to the node's left neighbor, and with probability 1/2 to the right neighbor.



Which vertex in G is the most likely to be the last one reached in this random walk? Justify your answer. (Clearly, it cannot be vertex #1, since it is reached initially.)

All vertices (excluding vertex #1) have the same probability of being the last one reached! To see why, note that the node i is the last one reached iff first one of its neighbors, $\{i-1, i+1\}$, is reached, and then the other neighbor is reached along the "long path" (i.e. the one not going through i) without visiting i). Let $t_j < t_k$ denote the event, that node j was reached before node k. Formally, we can write

$$\begin{aligned}
&\mathbf{Pr}[i \text{ reached as the last one}] = \\
&\quad = \mathbf{Pr}[t_{i-1} < t_{i+1}] \cdot \mathbf{Pr}[(i+1) \text{ reached from } (i-1) \text{ without touching } i \,|\, t_{i-1} < t_{i+1}] + \\
&\qquad \mathbf{Pr}[t_{i+1} < t_{i-1}] \cdot \mathbf{Pr}[(i-1) \text{ reached from } (i+1) \text{ without touching } i \,|\, t_{i+1} < t_{i-1}]
\end{aligned}$$

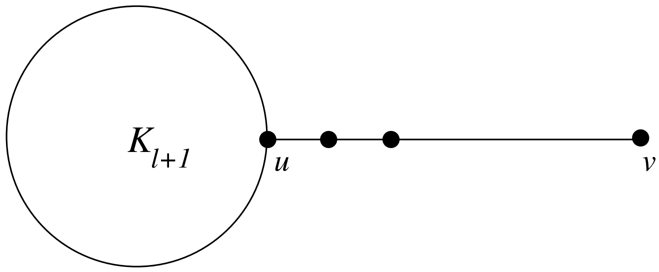
By symmetry, the conditional probabilities are equal, and independent of i, so denoting their value by p we get

$$\mathbf{Pr}[i \text{ reached as the last one}] = \underbrace{(\mathbf{Pr}[t_{i-1} < t_{i+1}] + \mathbf{Pr}[t_{i+1} < t_{i-1}])}_{=1} \cdot p = p\,.$$

Hence, since the above equation holds for all $i \neq 1$, the claim follows. Moreover, since one of the vertices has to be the last one, we get $p = 1/(n-1)$. □

Problem 2: Lollipop Graphs (30 points)

Consider a lollipop graph with $n = 2l + 1$ vertices (K_{l+1} denotes a complete graph with $l + 1$ vertices):



(a) Determine h_{uv} (hitting time) and c_{uv} (commute time) for u, v as marked on the figure, for $n = 5, 7, 2l + 1$, using the method of Markov Chains [Section 6.3 in Motwani&Raghavan]

(b) Same as part (a), but use the method of of Electric Networks [Section 6.4 in Motwani&Raghavan]

(Solution due to Jernej Barbic, Vincent Conitzer, Vahe Poladian)

(a)

We solve the general case for $n = 2l + 1$.

First, we note that because of the symmetry, the hitting time h_{av} is the same for any node a in the "interior" of K_{l+1}, i.e. for all nodes $a \in K_{l+1} \setminus \{u\}$. We have

$$h_{av} = \frac{l-1}{l}(h_{av} + 1) + \frac{1}{l}(h_{uv} + 1).$$

It follows from this equation that we have

$$h_{av} = l + h_{uv}.$$

Furthermore, we have

$$h_{uv} = \frac{l}{l+1}(h_{av} + 1) + \frac{1}{l+1}(h_{u_1,v} + 1),$$

where u_1 is the node to the right of u. Using the result for h_{av}, the last equation simplifies to

$$h_{uv} - h_{u_1,v} = l^2 + l + 1.$$

Let u_i be node which lies i steps to the right from u, i.e. $u = u_0, v = u_l$. We have for $i = 1, \ldots, l - 2$

$$h_{u_i,v} = \frac{1}{2}(h_{u_{i-1},v} + 1) + \frac{1}{2}(h_{u_{i+1},v} + 1) = 1 + \frac{1}{2}(h_{\text{left neighbour},v} + h_{\text{right neighbour},v}).$$

The last equation is

$$h_{u_1,v} = \frac{1}{2} \cdot 1 + \frac{1}{2}(1 + h_{u_2,v}).$$

Now, we can write a $l \times l$ system of equations

$$\begin{bmatrix} 1 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & & \cdots & & & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{bmatrix} \begin{bmatrix} h_{u_0,v} \\ h_{u_1,v} \\ h_{u_2,v} \\ \cdots \\ h_{u_{l-2},v} \\ h_{u_{l-1},v} \end{bmatrix} = \begin{bmatrix} l^2+l+1 \\ 2 \\ 2 \\ \cdots \\ 2 \\ 2 \end{bmatrix}$$

We are looking for $h_{uv} = h_{u_0,v}$. We can solve the system by Gaussian elimination: add first equation to second, then second to third and so on until the last equation. Now, add the last equation to the one before the last, then add the one before the last to the second one before the last and so on until you add the second equation to the first one. Now the system matrix is identity matrix. By performing operations corresponding to row operations on the right hand side vector, we obtain that the final value at position 1 is $l^2(l+2)$, which is our solution for h_{uv}.

Now, we must determine h_{vu}. In this case, we are actually determining the hitting time of the last node in the chain of length l, providing we start at the other side of the chain. Using the same technique as above, we now obtain the $l \times l$ linear system:

$$\begin{bmatrix} 1 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & & \cdots & & & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{bmatrix} \begin{bmatrix} h_{v,u_0} \\ h_{v,u_1} \\ h_{v,u_2} \\ \cdots \\ h_{v,u_{l-2}} \\ h_{v,u_{l-1}} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ \cdots \\ 2 \\ 2 \end{bmatrix}$$

Using Gaussian elimination (in the same way as above), we now obtain

$$h_{vu} = h_{v,u_0} = l^2.$$

Hence

$$c_{uv} = h_{uv} + h_{vu} = l^2(l+3).$$

(b)

For $n = 2l+1$, we have $m = \binom{l+1}{2} + l$. The resistance between u and v is l, as there is no other path from u to v other than the handle of the lollipop. It follows that

$$c_{uv} = 2ml = l^2(l+3).$$

We have $c_{uv} = h_{uv} + h_{vu}$. Hitting time h_{vu} is clearly the same as hitting time h'_{vu} in the graph (chain of length l) that consists just of the handle of the lollipop, i.e. lollipop with K_{l+1} removed (except u). Call this graph G'. Then

$$h_{vu} = h'_{vu} = h'_{uv} = \frac{1}{2}c'_{uv} = \frac{1}{2}2l \cdot l = l^2.$$

Now, we can compute

$$h_{uv} = c_{uv} - h_{vu} = l^2(l+2).$$

The results for h_{uv} and c_{uv} are of course the same as in (b).

Special cases

In both cases ((a) and (b)), the general formulas yield these results for the special cases:
$n = 5, l = 2: \quad c_{uv} = 20, \quad h_{uv} = 16$
$n = 7, l = 3: \quad c_{uv} = 54, \quad h_{uv} = 45$