Lecture 6

Foundations of Data Flow Analysis

I. Meet operator

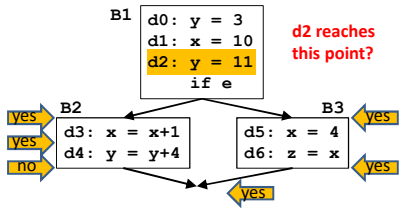
II. Transfer functions

III. Correctness, Precision, Convergence

IV. Efficiency

[ALSU 9.3]

Review: Reaching Definitions



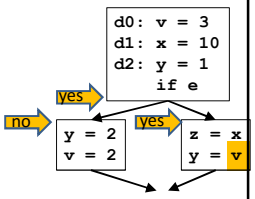
f	Gen	Kill
1	{1,2}	{3,4,5}
2	{3,4}	{1,2,5}
3	{5,6}	{1,3}

- a basic block b
  - **generates** definitions: Gen[b],
    - set of definitions in b that reach end of b
  - **kills** definitions: in[b] - Kill[b],
    where Kill[b]=set of defs (in rest of program) killed by defs in b
- **transfer function** f_b **: out[b] = Gen[b] U (in(b)-Kill[b])**
- **meet** operator:
  in[b] = out[p_1] U out[p_2] U ... U out[p_n], where
  p_1, ..., p_n are all predecessors of b

Review: Live Variable Analysis

- A variable **v** is **live** at point *p* if
  - the value of **v** is used along some path in the flow graph starting at *p*.
- **A basic block b can**
  - generate live variables: **Use[b]**
    - set of locally exposed uses in b
  - propagate incoming live variables: **OUT**[b] - **Def[b]**,
    - where Def[b]= set of variables defined in b.b.

v live at this point?



- **Backward analysis**
  - **transfer function** for block b:
    in[b] = Use[b] U (out(b)-Def[b])
- **meet** operator:
  out[b] = in[s_1] U in[s_2] U ... U in[s_n], where
  s_1, ..., s_n are all successors of b

Review: Data Flow Analysis Framework

	Reaching Definitions	Live Variables
Domain	Sets of definitions	Sets of variables
Direction	forward: out[b] = f_b(in[b]) in[b] = $\wedge$ out[pred(b)]	backward: in[b] = f_b(out[b]) out[b] = $\wedge$ in[succ(b)]
Transfer function	$f_b(x) = Gen_b \cup (x - Kill_b)$	$f_b(x) = Use_b \cup (x - Def_b)$
Meet Operation ($\wedge$)	$\cup$	$\cup$
Boundary Condition	out[entry] = $\varnothing$	in[exit] = $\varnothing$
Initial interior points	out[b] = $\varnothing$	in[b] = $\varnothing$

Other Data Flow Analysis problems fit into this general framework,
e.g., Available Expressions [ALSU 9.2.6]

A Unified Framework

- **Data flow problems are defined by**
  - Domain of values: V
  - Meet operator ($V \wedge V \rightarrow V$), initial value
  - A set of transfer functions ($V \rightarrow V$)
- **Usefulness of unified framework**
  - To answer questions such as correctness, precision, convergence, speed of convergence for a family of problems
    - If meet operators and transfer functions have properties X, then we know Y about the above.
  - Reuse code

Overview: A Check List for Data Flow Problems

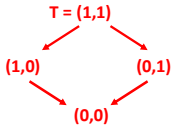
- **Semi-lattice**
  - set of values
  - meet operator
  - top, bottom
  - finite descending chain?
- **Transfer functions**
  - function of each basic block
  - monotone
  - distributive?
- **Algorithm**
  - initialization step (entry/exit, other nodes)
  - visit order: rPostOrder
  - depth of the graph

I. Meet Operator

- **Properties of the meet operator**
  - commutative: $x \wedge y = y \wedge x$

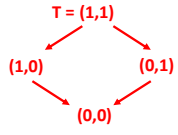
  - idempotent: $x \wedge x = x$
  - associative: $x \wedge (y \wedge z) = (x \wedge y) \wedge z$
  - there is a Top element T such that $x \wedge T = x$
- **Meet operator defines a partial ordering on values**
  - $x \leq y$ if and only if $x \wedge y = x$ [$y \rightarrow x$ in diagram]
    - Transitivity: if $x \leq y$ and $y \leq z$ then $x \leq z$
    - Antisymmetry: if $x \leq y$ and $y \leq x$ then $x = y$
    - Reflexivity: $x \leq x$

Partial Order

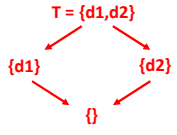


Meet Operator: Elementwise-min

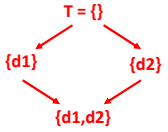
Partial Order



Meet Operator: Elementwise-min



Meet Operator: Intersection



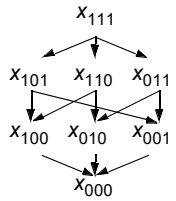
Meet Operator: Union

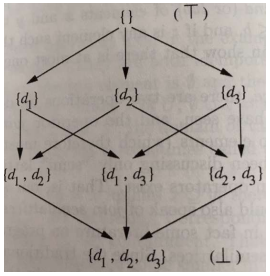
- Top and Bottom elements
  - Top T such that: $x \wedge T = x$
  - Bottom $\perp$ such that: $x \wedge \perp = \perp$
- Values and meet operator in a data flow problem define a semi-lattice:
  - there exists a T, but not necessarily a $\perp$.
- x, y are ordered: $x \leq y$ then $x \wedge y = x$ [$y \rightarrow x$ in diagram]
- what if x and y are not ordered?
  - $x \wedge y \leq x$, $x \wedge y \leq y$, and if $w \leq x$, $w \leq y$, then $w \leq x \wedge y$

One vs. All Variables/Definitions

- **Lattice for each variable: e.g. intersection**

- **Lattice for three variables:**





Descending Chain

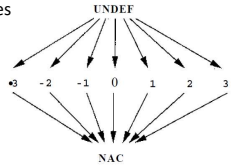
- **Definition**
  - The **height** of a lattice is the largest number of **> relations** that will fit in a descending chain.

$x_0 > x_1 > x_2 > \ldots$

- **Height of values in reaching definitions?**

  Height=n, where n is the number of definitions

- **Important property: finite descending chain**
- **Can an infinite lattice have a finite descending chain?** yes
- **Example: Constant Propagation/Folding**
  - To determine if a variable is a constant
- **Data values**
  - undef, ... -1, 0, 1, 2, ..., not-a-constant



II. Transfer Functions

- **Basic Properties** $f: V \rightarrow V$

out[b] = Gen[b] U (in(b)-Kill[b])

  - Has an identity function
    - There exists an f such that $f(x) = x$, for all x.
  - Closed under composition
    - if $f_1, f_2 \in F$, then $f_1 \cdot f_2 \in F$

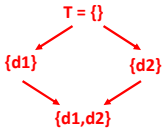
Monotonicity

out[b] = Gen[b] U (in(b)-Kill[b])

- A framework $(F, V, \wedge)$ is monotone if and only if
  - $x \leq y$ implies $f(x) \leq f(y)$
  - i.e. a "smaller or equal" input to the same function will always give a "smaller or equal" output
- Equivalently, a framework $(F, V, \wedge)$ is monotone if and only if
  - $f(x \wedge y) \leq f(x) \wedge f(y)$
  - i.e. merge input, then apply f is **small than or equal to** apply the transfer function individually and then merge the result

Example

- **Reaching definitions: f(x) = Gen ∪ (x - Kill), ∧ = ∪**
  - Definition 1:
    - $x_1 \leq x_2$, Gen ∪ (x_1 - Kill) ≤ Gen ∪ (x_2 - Kill)
  - Definition 2:
    - (Gen ∪ (x_1 - Kill)) ∪ (Gen ∪ (x_2 - Kill))
      = (Gen ∪ ((x_1 ∪ x_2) - Kill))



Meet Operator: Union

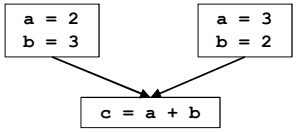
$[x_1 \leq x_2 \text{ iff } x_2 \rightarrow x_1]$

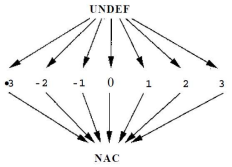
- **Note: Monotone framework does not mean that f(x) ≤ x**
  - e.g., reaching definition for two definitions in program
  - suppose: f_x: Gen_x = {d_1, d_2} ; $Kill_x$= {}
    then f(x) = {d_1, d_2} for any x, including x = {}

- **If input(second iteration) ≤ input(first iteration)**
  - result(second iteration) ≤ result(first iteration)

Distributivity

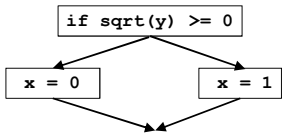
- A framework (*F, V, ∧*) is **distributive** if and only if
  - f(x ∧ y) = f(x) ∧ f(y)
  - i.e. merge input, then apply f is **equal to** apply the transfer function individually then merge result
- Example: Constant Propagation is NOT distributive





III. Data Flow Analysis

- **Definition**
  - Let $f_1, \ldots, f_m : \in F$, where f_i is the transfer function for node *i*
    - $f_p = f_{n_k} \cdot \ldots \cdot f_{n_1}$, where *p* is a path through nodes $n_1, \ldots, n_k$
    - f_p = identify function, if *p* is an empty path
- **Ideal data flow answer:**
  - For each node *n:*
    $\wedge f_{p_i}(\top)$, for all possibly executed paths p_i reaching *n*.

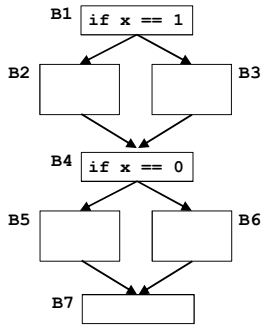


- **But: Determining all possibly executed paths is undecidable**

Meet-Over-Paths (MOP)

- **Err in the conservative direction**
- **Meet-Over-Paths (MOP):**
  - For each node *n:*
    MOP(*n*) = $\wedge f_{p_i}(\top)$, for all paths p_i reaching *n*
  - a path exists as long there is an edge in the code
  - consider more paths than necessary
  - MOP = Perfect-Solution ∧ Solution-to-Unexecuted-Paths
  - MOP ≤ Perfect-Solution
  - Potentially more constrained, solution is small
    - hence *conservative*
  - It is not **safe** to be > Perfect-Solution!
- **Desirable solution: as close to MOP as possible**

Example: MOP considers more paths than Ideal



Ideal: Considers only 2 paths
B1-B2-B4-B6-B7 (i.e., x=1)
B1-B3-B4-B5-B7 (i.e., x=0)

MOP: Also considers unexecuted paths
B1-B2-B4-B5-B7
B1-B3-B4-B6-B7

Assume: B2 & B3 do not update x

Solving Data Flow Equations

- **Example: Reaching definitions**
  - out[entry] = {}
  - Values = {subsets of definitions}
  - Meet operator: ∪
    - in[b] = ∪ out[*p*], for all predecessors *p* of b
  - Transfer functions: out[b] = $gen_b \cup (in[b] - kill_b)$
- **Any solution satisfying equations = Fixed Point Solution (FP)**
- **Iterative algorithm**
  - initializes out[b] to {}
  - if converges, then it computes Maximum Fixed Point (MFP):
    - MFP is the largest of all solutions to equations
- **Properties:**
  - FP ≤ MFP ≤ MOP ≤ Perfect-solution
  - FP, MFP are safe
  - in(b) ≤ MOP(b)

Partial Correctness of Algorithm

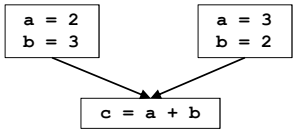
- **If data flow framework is monotone (i.e., x ≤ y implies f(x) ≤ f(y))**
  then if the algorithm converges, IN[b] ≤ MOP[b]
- **Proof: Induction on path lengths**
  - Define IN[entry] = OUT[entry]
    and transfer function of entry = Identity function
  - Base case: path of length 0
    - Proper initialization of IN[entry]
  - If true for path of length k, $p_k = (n_1, ..., n_k)$, then
    true for path of length k+1: $p_{k+1} = (n_1, ..., n_{k+1})$
    - Assume: $IN[n_k] \le f_{n_{k-1}}(f_{n_{k-2}}(...f_{n_1}(IN[entry])))$
    - $IN[n_{k+1}] = OUT[n_k] \wedge ...$

      $\le OUT[n_k] = f_{n_k}(IN[n_k])$

      $\le f_{n_k}(f_{n_{k-1}}(...f_{n_1}(IN[entry])))$ by inductive assumption & monotonicity

Precision

- **If data flow framework is distributive (i.e., f(x ∧ y) = f(x) ∧ f(y))**
  then if the algorithm converges, IN[b] = MOP[b]
- Monotone but not distributive: behaves as if there are additional paths

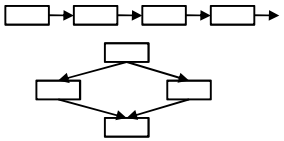


Additional Property to Guarantee Convergence

- **Data flow framework (monotone) converges if there is a finite descending chain**
- For each variable IN[b], OUT[b], consider the sequence of values set to each variable across iterations:
  - if sequence for in[b] is monotonically decreasing
    - sequence for out[b] is monotonically decreasing
      - (out[b] initialized to T)
  - if sequence for out[b] is monotonically decreasing
    - sequence of in[b] is monotonically decreasing

IV. Speed of Convergence

- Speed of convergence depends on order of node visits

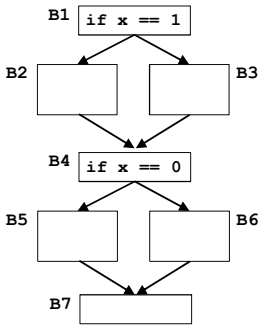


- Reverse "direction" for backward flow problems

Reverse Postorder

- **Step 1: depth-first post order**

```
main() {
    count = 1;
    Visit(root);
}
Visit(n) {
    for each successor s that
            has not been visited
        Visit(s);
    PostOrder(n) = count;
    count = count+1;
}
```



- **Step 2: reverse order**

```
For each node i
    rPostOrder(i)= NumNodes - PostOrder(i)
```


Depth-First Iterative Algorithm (forward)

```
input: control flow graph CFG = (N, E, Entry, Exit)
/* Initialize */
    out[entry] = init_value
    For all nodes i
        out[i] = T
    Change = True
/* iterate */
    While Change {
        Change = False
        For each node i in rPostOrder {
            in[i] = ∧(out[p]), for all predecessors p of i
            oldout = out[i]
            out[i] = f_i(in[i])
            if oldout ≠ out[i]
                Change = True
        }
    }
```

Speed of Convergence

- **If cycles do not add information**
  - information can flow in one pass down a series of nodes of increasing order number:
    - e.g., 1 -> 4 -> 5 -> 7 -> 2 -> 4 ...

      first pass
  - passes determined by number of back edges in the path
    - essentially the nesting depth of the graph
  - Number of iterations = number of back edges in any acyclic path + 2
    - (2 are necessary even if there are no cycles)
    - (2 not 1 since need a last pass where nothing changed)
- **What is the depth?**
  - corresponds to depth of intervals for "reducible" graphs
  - in real programs: average of 2.75 [ALSU 9.6.7]

Summary: A Check List for Data Flow Problems

- **Semi-lattice**
  - set of values
  - meet operator
  - top, bottom
  - finite descending chain?
- **Transfer functions**
  - function of each basic block
  - monotone
  - distributive?
- **Algorithm**
  - initialization step (entry/exit, other nodes)
  - visit order: rPostOrder
  - depth of the graph

Friday's Class

- Global common subexpression elimination
- Constant propagation/folding