Lecture 19

Array Dependence Analysis & Parallelization

[ALSU 11.6]

Data Dependence

$$
\begin{aligned}
S_1 &: \quad A = 1.0 \\
S_2 &: \quad B = A + 2.0 \\
S_3 &: \quad A = C - D \\
&\quad \vdots \\
S_4 &: \quad A = B/C
\end{aligned}
$$

We define four types of data dependence.

- Flow (true) dependence: a statement S_i precedes a statement S_j in execution and S_i computes a data value that S_j uses.
- Implies that S_i must execute before S_j.

$S_i \delta^t S_j$ $\quad (S_1 \delta^t S_2$ and $S_2 \delta^t S_4)$

Data Dependence

$$
\begin{aligned}
S_1 &: \quad A = 1.0 \\
S_2 &: \quad B = A + 2.0 \\
S_3 &: \quad A = C - D \\
&\quad \vdots \\
S_4 &: \quad A = B/C
\end{aligned}
$$

We define four types of data dependence.

- Anti dependence: a statement S_i precedes a statement S_j in execution and S_i uses a data value that S_j computes.
- It implies that S_i must be executed before S_j.

$S_i \delta^a S_j$ $\quad (S_2 \delta^a S_3)$

Data Dependence

$$
\begin{aligned}
S_1 &: \quad A = 1.0 \\
S_2 &: \quad B = A + 2.0 \\
S_3 &: \quad A = C - D \\
&\quad \vdots \\
S_4 &: \quad A = B/C
\end{aligned}
$$

We define four types of data dependence.

- Output dependence: a statement S_i precedes a statement S_j in execution and S_i computes a data value that S_j also computes.
- It implies that S_i must be executed before S_j.

$S_i \delta^o S_j$ $\quad (S_1 \delta^o S_3$ and $S_3 \delta^o S_4)$

Data Dependence

$$
\begin{aligned}
S_1 &: \quad A = 1.0 \\
S_2 &: \quad B = A + 2.0 \\
S_3 &: \quad A = C - D \\
&\quad\quad \vdots \\
S_4 &: \quad A = B/C
\end{aligned}
$$

We define four types of data dependence.

- Input dependence: a statement S_i precedes a statement S_j in execution and S_i uses a data value that S_j also uses.
- Does this imply that S_i must execute before S_j?

$S_i \delta^I S_j$ $\quad$ $(S_3 \delta^I S_4)$

Data Dependence (continued)

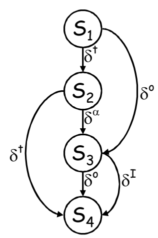
- The dependence is said to flow from S_i to S_j because S_i precedes S_j in execution.
- S_i is said to be the source of the dependence. S_j is said to be the sink of the dependence.
- The only "true" dependence is flow dependence; it represents the flow of data in the program.
- The other types of dependence are caused by programming style; they may be eliminated by re-naming.

$$
\begin{aligned}
S_1 &: \quad A = 1.0 \\
S_2 &: \quad B = A + 2.0 \\
S_3 &: \quad A1 = C - D \\
&\quad\quad \vdots \\
S_4 &: \quad A2 = B/C
\end{aligned}
$$

Data Dependence (continued)

- Data dependence in a program may be represented using a dependence graph G=(V,E), where the nodes V represent statements in the program and the directed edges E represent dependence relations.

$$
\begin{aligned}
S_1 &: \quad A = 1.0 \\
S_2 &: \quad B = A + 2.0 \\
S_3 &: \quad A = C - D \\
&\quad\quad \vdots \\
S_4 &: \quad A = B/C
\end{aligned}
$$



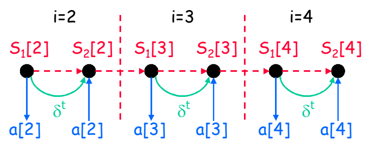
Value or Location?

- There are two ways a dependence is defined: value-oriented or location-oriented.

$$
\begin{aligned}
S_1 &: \quad A = 1.0 \\
S_2 &: \quad B = A + 2.0 \\
S_3 &: \quad A = C - D \\
&\quad\quad \vdots \\
S_4 &: \quad A = B/C
\end{aligned}
$$

Example 1

```
    for i = 2 to 4 {
S1:    a[i] = b[i] + c[i] ;
S2:    d[i] = a[i]
    }
```

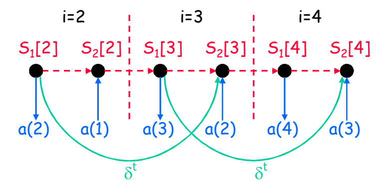


- There is an instance of S_1 that precedes an instance of S_2 in execution and S_1 produces data that S_2 consumes.
- S_1 is the source of the dependence; S_2 is the sink of the dependence.
- The dependence flows between instances of statements in the same iteration (loop-independent dependence).
- The number of iterations between source and sink (dependence distance) is 0. The dependence direction is =.

$S_1\, \delta^t_=\, S_2$ or $S_1\, \delta^t_0\, S_2$

Example 2

```
    do i = 2, 4
S1:    a(i) = b(i) + c(i)
S2:    d(i) = a(i-1)
    end do
```

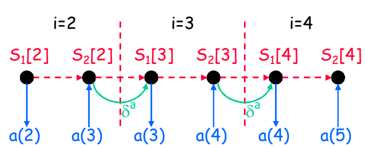


- There is an instance of S_1 that precedes an instance of S_2 in execution and S_1 produces data that S_2 consumes.
- S_1 is the source of the dependence; S_2 is the sink of the dependence.
- The dependence flows between instances of statements in different iterations (loop-carried dependence).
- The dependence distance is 1. The direction is positive (<).

$S_1\, \delta^t_<\, S_2$ or $S_1\, \delta^t_1\, S_2$

Example 3

```
    do i = 2, 4
S1:    a(i) = b(i) + c(i)
S2:    d(i) = a(i+1)
    end do
```



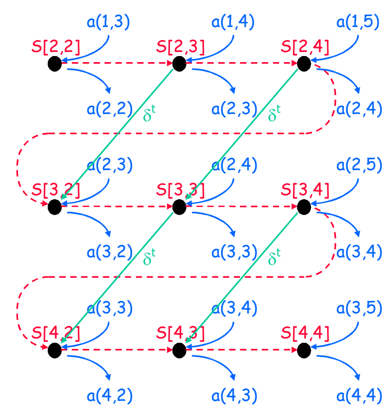
- There is an instance of S_2 that precedes an instance of S_1 in execution and S_2 consumes data that S_1 produces.
- S_2 is the source of the dependence; S_1 is the sink of the dependence.
- The dependence is loop-carried.
- The dependence distance is 1.

$S_2\, \delta^a_<\, S_1$ or $S_2\, \delta^a_1\, S_1$

- Are you sure you know why it is $S_2\, \delta^a_<\, S_1$ even though S_1 appears before S_2 in the code?

Example 4

```
   do i = 2, 4
      do j = 2, 4
S:       a(i,j) = a(i-1,j+1)
      end do
   end do
```



- An instance of S precedes another instance of S and S produces data that S consumes.
- S is both source and sink.
- The dependence is loop-carried.
- The dependence distance is (1,-1).

$S\, \delta^t_{(<,>)}\, S$ or $S\, \delta^t_{(1,-1)}\, S$

Problem Formulation

- Consider the following perfect nest of depth d:

$$
\begin{array}{l}
\text{do } I_1 = L_1, U_1 \\
\quad \text{do } I_2 = L_2, U_2 \\
\qquad \ddots \\
\qquad \text{do } I_d = L_d, U_d \\
\qquad\quad a(f_1(\vec{I}), f_2(\vec{I}), \cdots, f_m(\vec{I})) = \cdots \\
\qquad\quad \cdots = a(g_1(\vec{I}), g_2(\vec{I}), \cdots, g_m(\vec{I})) \\
\qquad \text{enddo} \\
\qquad \cdot^{\cdot^{\cdot}} \\
\quad \text{enddo} \\
\text{enddo}
\end{array}
$$

array reference: $a(\;, f_k(\vec{I}), \cdots, \;)$

subscript position

subscript function or subscript expression

$\vec{I} = (I_1, I_2, \cdots, I_d)$

$\vec{L} = (L_1, L_2, \cdots, L_d)$

$\vec{U} = (U_1, U_2, \cdots, U_d)$

$\vec{L} \le \vec{U}$

linear functions
$b_0 + b_1 I_1 + b_2 I_2 + \cdots + b_d I_d$

Problem Formulation

- Dependence will exist if there exists two iteration vectors $\vec{k}$ and $\vec{j}$ such that $\vec{L} \le \vec{k} \le \vec{j} \le \vec{U}$ and:

$$
\begin{array}{ll}
 & f_1(\vec{k}) = g_1(\vec{j}) \\
\text{and} & f_2(\vec{k}) = g_2(\vec{j}) \\
\text{and} & \vdots \\
\text{and} & f_m(\vec{k}) = g_m(\vec{j})
\end{array}
$$

- That is:

$$
\begin{array}{ll}
 & f_1(\vec{k}) - g_1(\vec{j}) = 0 \\
\text{and} & f_2(\vec{k}) - g_2(\vec{j}) = 0 \\
\text{and} & \vdots \\
\text{and} & f_m(\vec{k}) - g_m(\vec{j}) = 0
\end{array}
$$

Problem Formulation - Example

```
     do i = 2, 4
S1:    a(i) = b(i) + c(i)
S2:    d(i) = a(i-1)
     end do
```

- Does there exist two iteration vectors i_1 and i_2, such that $2 \le i_1 \le i_2 \le 4$ and such that:

  $i_1 = i_2 - 1$?

- Answer: yes; i_1=2 & i_2=3 and i_1=3 & i_2 =4.
- Hence, there is dependence!
- The dependence distance vector is i_2-i_1 = 1.
- The dependence direction vector is sign(1) = <.

Problem Formulation - Example

```
     do i = 2, 4
S1:    a(i) = b(i) + c(i)
S2:    d(i) = a(i+1)
     end do
```

- Does there exist two iteration vectors i_1 and i_2, such that $2 \le i_1 \le i_2 \le 4$ and such that:

  $i_1 = i_2 + 1$?

- Answer: yes; i_1=3 & i_2=2 and i_1=4 & i_2 =3. (But, but!).
- Hence, there is dependence!
- The dependence distance vector is i_2-i_1 = -1.
- The dependence direction vector is sign(-1) = >.
- Is this possible?

Problem Formulation - Example

```
      do i = 1, 10
S1:     a(2*i) = b(i) + c(i)
S2:     d(i) = a(2*i+1)
      end do
```

- Does there exist two iteration vectors i_1 and i_2, such that $1 \le i_1 \le i_2 \le 10$ and such that:

  $2*i_1 = 2*i_2 + 1$?

- Answer: no; $2*i_1$ is even & $2*i_2+1$ is odd.
- Hence, there is no dependence!

Problem Formulation

- Dependence testing is equivalent to an integer linear programming (ILP) problem of 2d variables & m+d constraint!
- An algorithm that determines if there exists two iteration vectors $\vec{k}$ and $\vec{j}$ that satisfies these constraints is called a dependence tester.

$$
\begin{array}{l}
\text{do } I_1 = L_1, U_1 \\
\quad \text{do } I_2 = L_2, U_2 \\
\qquad \ddots \\
\qquad\quad \text{do } I_d = L_d, U_d \\
\qquad\qquad a(f_1(\vec{I}), f_2(\vec{I}), \cdots, f_m(\vec{I})) = \cdots \\
\qquad\qquad \cdots = a(g_1(\vec{I}), g_2(\vec{I}), \cdots, g_m(\vec{I})) \\
\qquad\quad \text{enddo} \\
\qquad \iddots \\
\quad \text{enddo} \\
\text{enddo}
\end{array}
$$

Problem Formulation

- Dependence testing is equivalent to an integer linear programming (ILP) problem of 2d variables & m+d constraint!
- An algorithm that determines if there exists two iteration vectors $\vec{k}$ and $\vec{j}$ that satisfies these constraints is called a dependence tester.
- The dependence distance vector is given by $\vec{j} - \vec{k}$.
- The dependence direction vector is give by sign($\vec{j} - \vec{k}$).
- Dependence testing is NP-complete!
- A dependence test that reports dependence only when there is dependence is said to be exact. Otherwise it is in-exact.
- A dependence test must be conservative; if the existence of dependence cannot be ascertained, dependence must be assumed.

Dependence Testers

- Lamport's Test.
- GCD Test.
- Banerjee's Inequalities.
- Generalized GCD Test.
- Power Test.
- I-Test.
- Omega Test.
- Delta Test.
- Stanford Test.
- etc…

Lamport's Test

- Lamport's Test is used when there is a single index variable in the subscript expressions, and when the coefficients of the index variable in both expressions are the same.

$$A(\cdots, b*i + c_1, \cdots) = \cdots$$
$$\cdots = A(\cdots, b*i + c_2, \cdots)$$

- The dependence problem: does there exist i_1 and i_2, such that $L_i \le i_1 \le i_2 \le U_i$ and such that

$$b*i_1 + c_1 = b*i_2 + c_2? \quad \text{or} \quad i_2 - i_1 = \frac{c_1 - c_2}{b}?$$

- There is integer solution if and only if $\frac{c_1 - c_2}{b}$ is integer.
- The dependence distance is $d = \frac{c_1 - c_2}{b}$ if $L_i \le |d| \le U_i$.
- $d > 0 \Rightarrow$ true dependence.
  $d = 0 \Rightarrow$ loop independent dependence.
  $d < 0 \Rightarrow$ anti dependence.

Lamport's Test - Example

```
   do i = 1, n
     do j = 1, n
S:     a(i,j) = a(i-1,j+1)
     end do
   end do
```

- $i_1 = i_2 - 1$?

  $b = 1$; $c_1 = 0$; $c_2 = -1$

  $\frac{c_1 - c_2}{b} = 1$

  There is dependence.

  Distance (i) is 1.

- $j_1 = j_2 + 1$?

  $b = 1$; $c_1 = 0$; $c_2 = 1$

  $\frac{c_1 - c_2}{b} = -1$

  There is dependence.

  Distance (j) is -1.

$S\delta^t_{(1,-1)}S$ or $S\delta^t_{(<,>)}S$

Lamport's Test - Example

```
   do i = 1, n
     do j = 1, n
S:     a(i,2*j) = a(i-1,2*j+1)
     end do
   end do
```

- $i_1 = i_2 - 1$?

  $b = 1$; $c_1 = 0$; $c_2 = -1$

  $\frac{c_1 - c_2}{b} = 1$

  There is dependence.

  Distance (i) is 1.

- $2*j_1 = 2*j_2 + 1$?

  $b = 2$; $c_1 = 0$; $c_2 = 1$

  $\frac{c_1 - c_2}{b} = -\frac{1}{2}$

  There is no dependence.

?

There is no dependence!

GCD Test

- Given the following equation:

  $\sum_{\{i=1\}}^{n} a_i x_i = c$ where a_i and c are integers

  an integer solution exists if and only if:

  $\gcd(a_1, a_2, \ldots, a_n)$ divides c

- Problems:
  - ignores loop bounds
  - gives no information on distance or direction of dependence
  - often gcd(……) is 1 which always divides c, resulting in false dependences

GCD Test - Example

```
    do i = 1, 10
S1:   a(2*i) = b(i) + c(i)
S2:   d(i) = a(2*i-1)
    end do
```

- Does there exist two iteration vectors i_1 and i_2, such that $1 \leq i_1 \leq i_2 \leq 10$ and such that:

  $2*i_1 = 2*i_2 - 1$?

  or

  $2*i_2 - 2*i_1 = 1$?
- There will be an integer solution if and only if gcd(2,-2) divides 1.
- This is not the case, and hence, there is no dependence!

GCD Test Example

```
    do i = 1, 10
S1:   a(i) = b(i) + c(i)
S2:   d(i) = a(i-100)
    end do
```

- Does there exist two iteration vectors i_1 and i_2, such that $1 \leq i_1 \leq i_2 \leq 10$ and such that:

  $i_1 = i_2 - 100$?

  or

  $i_2 - i_1 = 100$?
- There will be an integer solution if and only if gcd(1,-1) divides 100.
- This is the case, and hence, there is dependence! Or is there?

Dependence Testing Complications

- Unknown loop bounds:

```
    do i = 1, N
S1:   a(i) = a(i+10)
    end do
```

  What is the relationship between N and 10?

- Triangular loops:

```
    do i = 1, N
      do j = 1, i-1
S:      a(i,j) = a(j,i)
      end do
    end do
```

  Must impose $j < i$ as an additional constraint.

More Complications

- User variables:

```
    do i = 1, 10
S1:   a(i) = a(i+k)
    end do
```

  Same problem as unknown loop bounds, but occur due to some loop transformations (e.g., loop bounds normalization).

```
    do i = L, H
S1:   a(i) = a(i-1)
    end do
```

$\Downarrow$

```
    do i = 1, H-L
S1:   a(i+L) = a(i+L-1)
    end do
```

More Complications

- Scalars:

```
     do i = 1, N                 do i = 1, N
S1:    x = a(i)        ⇒    S1:    x(i) = a(i)
S2:    b(i) = x             S2:    b(i) = x(i)
     end do                      end do
```

```
     j = N-1
     do i = 1, N                 do i = 1, N
S1:    a(i) = a(j)     ⇒    S1:    a(i) = a(N-i)
S2:    j = j - 1
     end do                      end do
```

```
     sum = 0
     do i = 1, N                 do i = 1, N
S1:    sum = sum + a(i) ⇒   S1:    sum(i) = a(i)
     end do                      end do
                                 sum += sum(i)   i = 1, N
```


Loop Parallelization

- A dependence is said to be carried by a loop if the loop is the outermost loop whose removal eliminates the dependence. If a dependence is not carried by the loop, it is loop-independent.

```
do i = 2, n-1
  do j = 2, m-1
    a(i, j) = …
    …         = a(i, j)

    b(i, j) = …
    …    = b(i, j-1)

    c(i, j) = …
    …    = c(i-1, j)
  end do
end do
```


Loop Parallelization

- A dependence is said to be carried by a loop if the loop is the outermost loop whose removal eliminates the dependence. If a dependence is not carried by the loop, it is loop-independent.

```
           do i = 2, n-1
             do j = 2, m-1
δt(=,=)        a(i, j) = …
               …         = a(i, j)

δt(=,<)        b(i, j) = …
               …    = b(i, j-1)

δt(<,=)        c(i, j) = …
               …    = c(i-1, j)
             end do
           end do
```

- Outermost loop with a non "=" direction carries dependence!

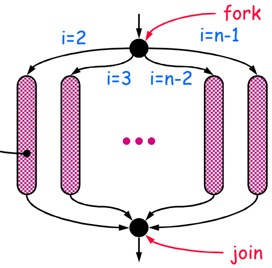
Loop Parallelization

The iterations of a loop may be executed in parallel with one another if and only if no dependences are carried by the loop!

Loop Parallelization - Example

$\delta^t_{=,<}$

```
do i = 2, n-1
  do j = 2, m-1
    b(i, j) = ...
    ...     = b(i, j-1)
  end do
end do
```

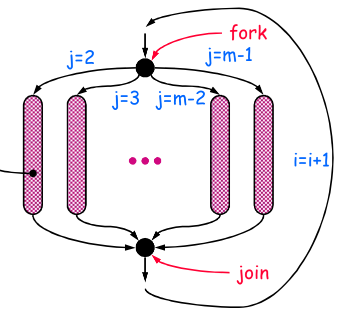


- Iterations of loop j must be executed sequentially, but the iterations of loop i may be executed in parallel.
- Outer loop parallelism.

Loop Parallelization - Example

$\delta^t_{<,=}$

```
do i = 2, n-1
  do j = 2, m-1
    b(i, j) = ...
    ...     = b(i-1, j)
  end do
end do
```

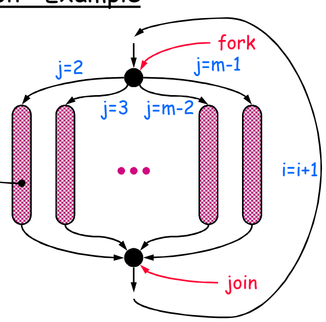


- Iterations of loop i must be executed sequentially, but the iterations of loop j may be executed in parallel.
- Inner loop parallelism.

Loop Parallelization - Example

$\delta^t_{<,<}$

```
do i = 2, n-1
  do j = 2, m-1
    b(i, j) = ...
    ...     = b(i-1, j-1)
  end do
end do
```

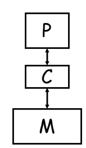


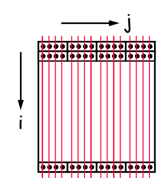
- Iterations of loop i must be executed sequentially, but the iterations of loop j may be executed in parallel. Why?
- Inner loop parallelism.

Loop Interchange

Loop interchange changes the order of the loops to improve the spatial locality of a program.

```
do j = 1, n
  do i = 1, n
    ... a(i,j) ...
  end do
end do
```



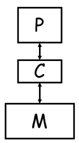


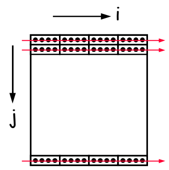
Loop Interchange

Loop interchange changes the order of the loops to improve the spatial locality of a program.

```
do j = 1, n
  do i = 1, n
     ... a(i,j) ...
  end do
end do
```

```
do i = 1, n
  do j = 1, n
     ... a(i,j) ...
  end do
end do
```





Loop Interchange

- Loop interchange can improve the granularity of parallelism!

```
do i = 1, n
  do j = 1, n
     a(i,j) = b(i,j)
     c(i,j) = a(i-1,j)
  end do
end do
```

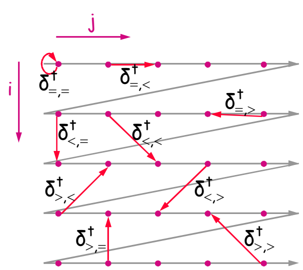
$\delta^{t}_{<,=}$

```
do j = 1, n
  do i = 1, n
     a(i,j) = b(i,j)
     c(i,j) = a(i-1,j)
  end do
end do
```

$\delta^{t}_{=,<}$

Loop Interchange

```
do i = 1,n
  do j = 1,n
    ... a(i,j) ...
  end do
end do
```

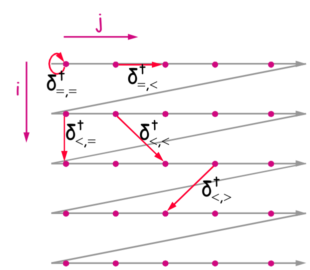



```
do j = 1,n
  do i = 1,n
    ... a(i,j) ...
  end do
end do
```

- When is loop interchange legal?

Loop Interchange

```
do i = 1,n
  do j = 1,n
    ... a(i,j) ...
  end do
end do
```

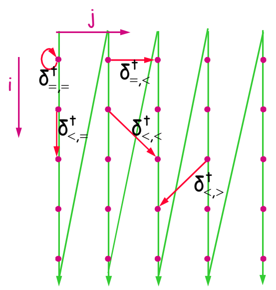



```
do j = 1,n
  do i = 1,n
    ... a(i,j) ...
  end do
end do
```

- When is loop interchange legal?

Loop Interchange

```
do i = 1,n
  do j = 1,n
     ... a(i,j) ...
  end do
end do
```

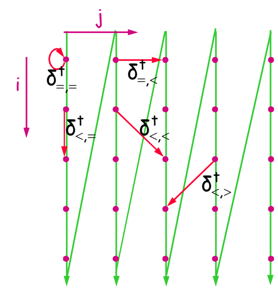



```
do j = 1,n
  do i = 1,n
     ... a(i,j) ...
  end do
end do
```

- When is loop interchange legal?

Loop Interchange

```
do i = 1,n
  do j = 1,n
     ... a(i,j) ...
  end do
end do
```




```
do j = 1,n
  do i = 1,n
     ... a(i,j) ...
  end do
end do
```

- When is loop interchange legal? when the "interchanged" dependences remain lexiographically positive!

Loop Blocking (Tiling)

```
do t = 1,T
  do i = 1,n
    do j = 1,n
       ... a(i,j) ...
    end do
  end do
end do
```

```
do t = 1,T
 do ic = 1, n, B
  do i = 1,B
    do jc = 1, n, B
     do j = 1,B
        ... a(ic+i-1,jc+j-1) ...
     end do
  end do
 end do
end do
```

```
do ic = 1, n, B
  do jc = 1, n , B
    do t = 1,T
      do i = 1,B
        do j = 1,B
          ... a(ic+i-1,jc+j-1) ...
        end do
      end do
    end do
  end do
end do
```

- When is loop blocking legal?

Wednesday's Class

- Global Scheduling, Software Pipelining [ALSU 10.4 - 10.5]