Lecture 15: Linear Programming III

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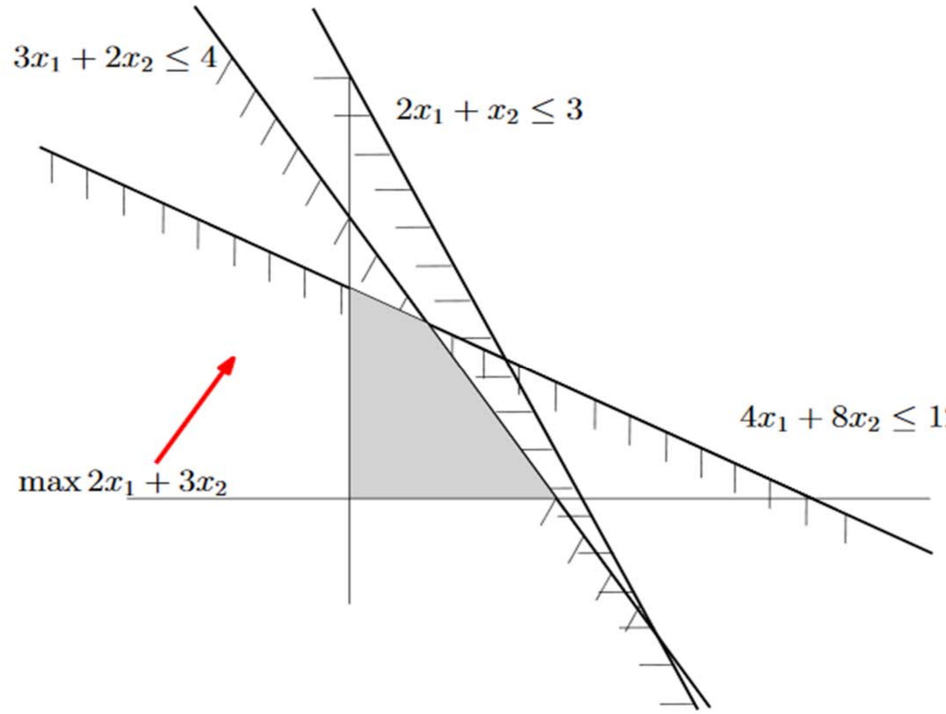
Outline

- Linear Programming Duality

- (Time permitting) More on the Ellipsoid Algorithm

$$P = \max(2x_1 + 3x_2)$$
$$\text{s.t.} \quad 4x_1 + 8x_2 \le 12$$
$$2x_1 + x_2 \le 3$$
$$3x_1 + 2x_2 \le 4$$
$$x_1, x_2 \ge 0$$



Since $2x_1 + 3x_2 \le 4x_1 + 8x_2 \le 12$, we know OPT ≤ 12

Since $2x_1 + 3x_2 \le \frac{1}{2}(4x_1 + 8x_2) \le 6$, we know OPT ≤ 6

Since $2x_1 + 3x_2 \le \frac{1}{3}\big((4x_1 + 8x_2) + (2x_1 + x_2)\big) \le 5$, we know OPT ≤ 5

Duality

- We took non-negative linear combinations of the constraints
- How do we find the best upper bound on OPT this way?
- Let $y_1, y_2, y_3 \geq 0$ be the coefficients of our linear combination. Then,

$$4y_1 + 2y_2 + 3y_3 \geq 2$$
$$8y_1 + y_2 + 2y_3 \geq 3$$
$$y_1, y_2, y_3 \geq 0$$

and we seek $\min(12y_1 + 3y_2 + 4y_3)$

$$P = \max(2x_1 + 3x_2)$$
$$\text{s.t.} \quad 4x_1 + 8x_2 \leq 12$$
$$2x_1 + x_2 \leq 3$$
$$3x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Primal LP

$$P = \max(2x_1 + 3x_2)$$
$$\text{s.t.} \quad 4x_1 + 8x_2 \leq 12$$
$$2x_1 + x_2 \leq 3$$
$$3x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Dual LP

$$4y_1 + 2y_2 + 3y_3 \geq 2$$
$$8y_1 + y_2 + 2y_3 \geq 3$$
$$y_1, y_2, y_3 \geq 0$$
$$\text{and we seek } \min(12y_1 + 3y_2 + 4y_3)$$

- If (x_1, x_2) is feasible for the primal, and (y_1, y_2, y_3) feasible for the dual,
$$2x_1 + 3x_2 \leq 12y_1 + 3y_2 + 4y_3$$
- If these are equal, we've found the optimal value for both LPs
- $(x_1, x_2) = (\frac{1}{2}, \frac{5}{4})$ and $(y_1, y_2, y_3) = (\frac{5}{16}, 0, \frac{1}{4})$ give the same value 4.75, so optimal

Dual LP

$$4y_1 + 2y_2 + 3y_3 \geq 2$$
$$8y_1 + y_2 + 2y_3 \geq 3$$
$$y_1, y_2, y_3 \geq 0$$

and we seek $\min(12y_1 + 3y_2 + 4y_3)$

- Let's try do the same thing to the dual:
- $12y_1 + 3y_2 + 4y_3 \geq 4y_1 + 2y_2 + 3y_2 \geq 2$
- $12y_1 + 3y_2 + 4y_3 \geq 8y_1 + y_2 + 2y_3 \geq 3$
- $12y_1 + 3y_2 + 4y_3 \geq \frac{2}{3}(4y_1 + 2y_2 + 3y_2) + (8y_1 + y_2 + 2y_3) \geq \frac{4}{3} + 3$

Dual LP

$$4y_1 + 2y_2 + 3y_3 \geq 2$$
$$8y_1 + y_2 + 2y_3 \geq 3$$
$$y_1, y_2, y_3 \geq 0$$
$$\text{and we seek } \min(12y_1 + 3y_2 + 4y_3)$$

$$P = \max(2x_1 + 3x_2)$$
$$\text{s.t.} \quad 4x_1 + 8x_2 \leq 12$$
$$2x_1 + x_2 \leq 3$$
$$3x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

- Take non-negative linear combination of the two constraints
- How do we find the best lower bound on OPT this way?
- Let $x_1, x_2 \geq 0$ be the coefficients of our linear combination. Then,
- $4x_1 + 8x_2 \leq 12$, $2x_1 + x_2 \leq 3$, $3x_1 + 2x_2 \leq 4$, $x_1 \geq 0$, $x_2 \geq 0$ and we seek to maximize $2x_1 + 3x_2$

We got back the primal!

Exercise: Consider the "primal" LP below on the left:

$$
\begin{aligned}
P = \max(7x_1 - x_2 + 5x_3) \\
\text{s.t.} \quad x_1 + x_2 + 4x_3 \le 8 \\
3x_1 - x_2 + 2x_3 \le 3 \\
2x_1 + 5x_2 - x_3 \le -7 \\
x_1, x_2, x_3 \ge 0
\end{aligned}
\qquad\qquad
\begin{aligned}
D = \min(8y_1 + 3y_2 - 7y_3) \\
\text{s.t.} \quad y_1 + 3y_2 + 2y_3 \ge 7 \\
y_1 - y_2 + 5y_3 \ge -1 \\
4y_1 + 2y_2 - y_3 \ge 5 \\
y_1, y_2, y_3 \ge 0
\end{aligned}
$$

Show that the problem of finding the best upper bound obtained using linear combinations of the constraints can be written as the LP above on the right (the "dual" LP). Also, now formulate the problem of finding a lower bound for the dual LP. Show this lower-bounding LP is just the primal (P).

Non-Nice Constraints

$$P = \max(7x_1 - x_2 + 5x_3)$$

$$\text{s.t.} \quad x_1 + x_2 + 4x_3 \leq 8$$

$$3x_1 - x_2 + 2x_3 \geq 3$$

$$x_1, x_2, x_3 \geq 0$$

$$D = \min(8y_1 + 3y_2)$$

$$\text{s.t.} \quad y_1 + 3y_2 \geq 7$$

$$y_1 - y_2 \geq -1$$

$$4y_1 + 2y_2 \geq 5$$

$$y_1 \geq 0, y_2 \leq 0$$

Formal Definition of Duality

Primal

Max $c^T x$

subject to $Ax \leq b$

$x \geq 0$

Dual

Min $b^T y$

subject to $A^T y \geq c$

$y \geq 0$

- Dual of the dual is the primal!
- Can we get better upper/lower bounds by looking at more complicated combinations of the inequalities, not just linear combinations?

Weak Duality

Primal

Max $c^T x$

subject to $Ax \leq b$

$x \geq 0$

Dual

Min $b^T y$

subject to $A^T y \geq c$

$y \geq 0$

- (Weak Duality) If x is a feasible solution of the primal, and y is a feasible solution of the dual, then $c^T x \leq b^T y$

- Proof: Since $x \geq 0$ and $y \geq 0$,

$$c^T x \leq y^T A x \leq y^T b = b^T y$$

Strong Duality

Primal

Max $c^T x$

subject to $Ax \leq b$

$x \geq 0$

Dual

Min $b^T y$

subject to $A^T y \geq c$

$y \geq 0$

- (Strong Duality) If primal is feasible and bounded (i.e., optimal value is not ∞), then dual is feasible and bounded. If x^* is optimal solution to the primal, and y^* is optimal solution to dual, then

$$c^T x^* = b^T y^*$$

- To prove x^* is optimal, I can give you y^* and you can check if x^* is feasible for the primal, y^* is feasible for the dual, and $c^T x^* = b^T y^*$

Consequences of Duality

$P \backslash D$	I	O	U
I	?	?	?
O	?	?	?
U	?	?	?

I means infeasible
O means feasible and bounded
U means unbounded

Which combinations are possible?

Consequences of Duality

$P \backslash D$	I	O	U
I	✓	X	✓
O	X	✓	X
U	✓	X	X

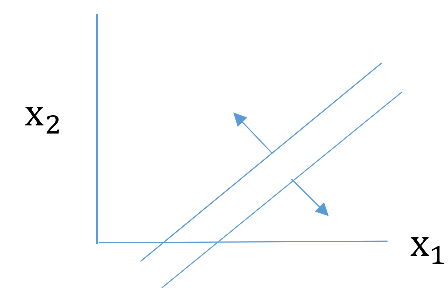
I means infeasible
O means feasible and bounded
U means unbounded

Check means possible
X means impossible

Possible Scenarios

$P\backslash D$	I	O	U
I	✓	X	✓
O	X	✓	X
U	✓	X	X

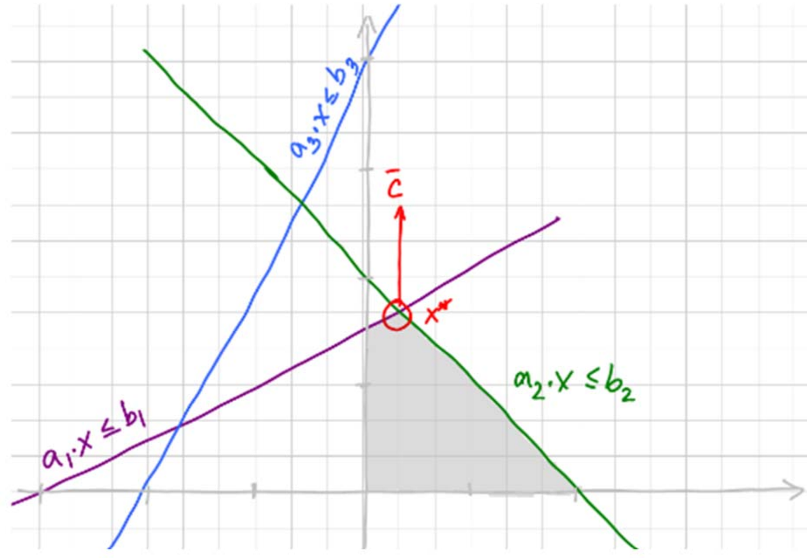
- Suppose primal is feasible and bounded
- By strong duality, dual is feasible and bounded
- If primal (maximization) is unbounded, by weak duality, $c^T x \le b^T y$, so no feasible dual solution e.g., $\max x_1$ subject to $x_1 \ge 1$ and $x_1 \ge 0$



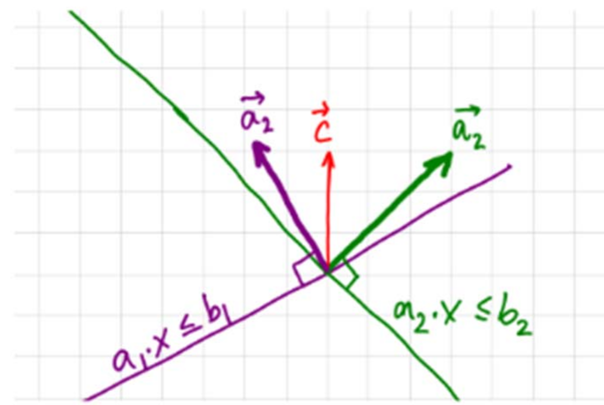
- Can primal and dual both be infeasible?
- Primal: max $2x_1 - x_2$ subject to $x_1 - x_2 \le 1$ and $-x_1 + x_2 \le -2$ and $x_1 \ge 0, x_2 \ge 0$
- Dual: $y_1 \ge 0,\ y_2 \ge 0$, and $y_1 - y_2 \ge 2$ and $-y_1 + y_2 \ge -1$, and $\min y_1 - 2y_2$
- Constraints are same for primal and dual, and both infeasible

Strong Duality Intuition

$$\begin{aligned} \text{maximize } & x_2 \\ \text{subject to } & -x_1 + 2x_2 \leq 3 \\ & x_1 + x_2 \leq 2 \\ & -2x_1 + x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$



$$a_1 = (-1,2), b_1 = 3$$
$$a_2 = (1,1), b_2 = 2$$
$$a_3 = (-2,1), b_3 = 4$$



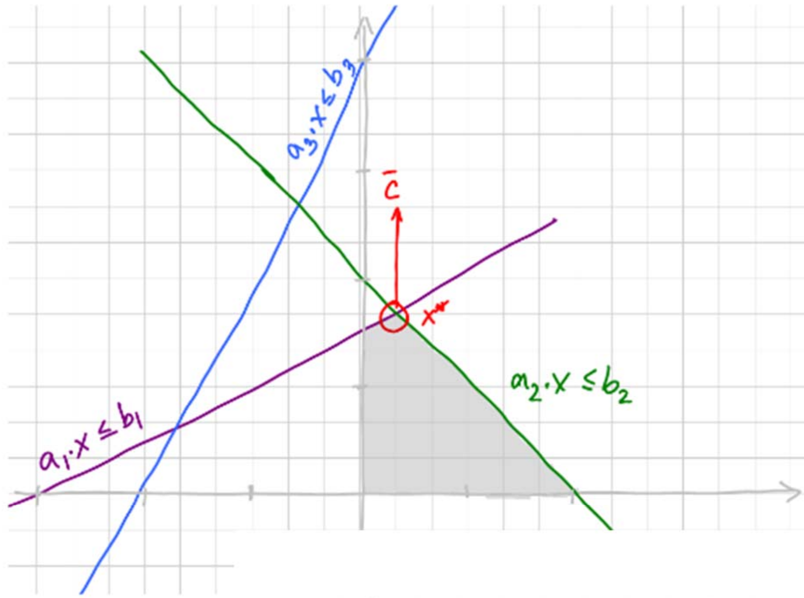
x^* satisfies $a_1 x = b_1$ and $a_2 x = b_2$

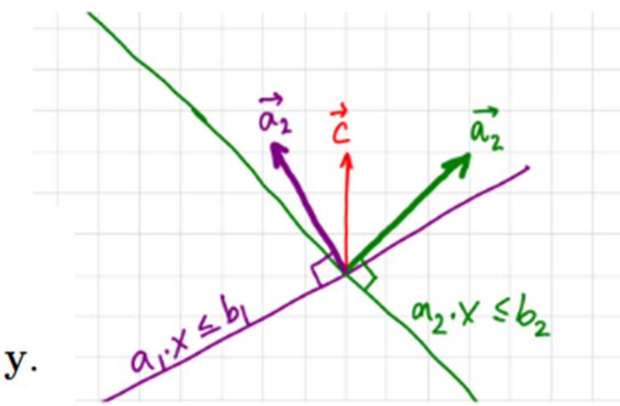
Strong Duality Intuition

- For non-negative y_1 and y_2

$$\mathbf{c} = y_1\mathbf{a}_1 + y_2\mathbf{a}_2.$$

$$\begin{aligned}\mathbf{c}^T \cdot \mathbf{x}^* &= (y_1\,\mathbf{a}_1 + y_2\,\mathbf{a}_2)\cdot\mathbf{x}^* \\ &= y_1(\mathbf{a}_1\cdot\mathbf{x}^*) + y_2(\mathbf{a}_2\cdot\mathbf{x}^*) \\ &= y_1 b_1 + y_2 b_2\end{aligned}$$





Defining $\mathbf{y} = (y_1, y_2, 0, \ldots, 0)$, we get

$$\text{optimal value of primal} = \mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y} = \text{value of dual solution } \mathbf{y}.$$

the $\mathbf{y}$ we found satisfies $\mathbf{c} = y_1\mathbf{a}_1 + y_2\mathbf{a}_2 = \sum_i y_i\mathbf{a}_i = A^T\mathbf{y}$, and hence $\mathbf{y}$ satisfies the dual constraints $\mathbf{y}^T A \ge \mathbf{c}^T$ by construction.

But $\mathrm{b}^T\mathrm{y} \ge \mathrm{c}^T\mathrm{x}^*$ by weak duality, so y is optimal!

Duality in Zero-Sum Games

- R is an n x m row payoff matrix
- W.l.o.g. R has all non-negative entries
- Variables: $v, p_1, \ldots, p_n$
- Max v

  subject to $p_i \geq 0$ for all rows i, $\sum_i p_i = 1$, $\sum_i p_i R_{i,j} \geq v$ for all columns j

- Replace $\sum_i p_i = 1$ with $\sum_i p_i \leq 1$.
- Include $v \geq 0$
- Write $\sum_i p_i R_{i,j} \geq v$ as $v - \sum_i p_i R_{i,j} \leq 0$

Duality in Zero-Sum Games

max $c^T x$ subject to $Ax \leq b$ and $x \geq 0$

$$\mathbf{x} = \begin{bmatrix} v \\ p_1 \\ p_2 \\ \dots \\ p_n \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \dots \\ 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{bmatrix}, \text{and } A = \begin{bmatrix} 1 & \\ 1 & -R^T \\ \dots & \\ 1 & \\ 0 & 1 \quad \dots \quad 1 \end{bmatrix}.$$

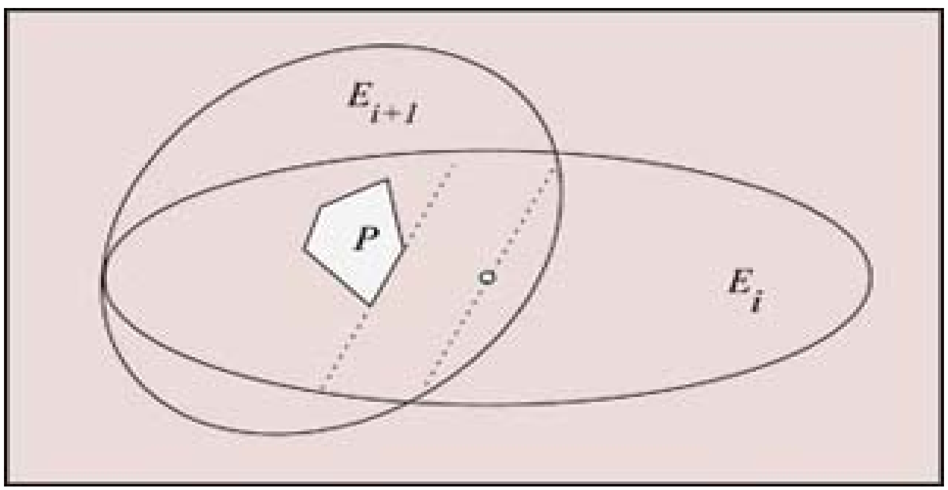
- Dual: min $y^T b$ subject to $y^T A \geq c^T$ and $y \geq 0$ for $y = (y_1, \ldots, y_{m+1})$
- Dual constraints say $y_1 + \cdots + y_m \geq 1$ and $\sum_j y_j R_{ij} \leq y_{m+1}$ for all rows i
  - Since we're minimizing y_{m+1} and $R_{i,j}$ all non-negative, $y_1 + \ldots + y_m = 1$
- y_{m+1} is value to the row player and $y_1, \ldots, y_m$ is column player's strategy
- Strong duality: $\max_p \min_j \sum_i p_i R_{ij} = \min_{y_1,\ldots,y_m} \max_i \sum_j y_j R_{ij}$

Ellipsoid Algorithm

Solves feasibility problem

Replace objective function with constraint, do binary search

Replace “minimize $x_1 + x_2$" with $x_1 + x_2 \le \lambda$



Can handle exponential number of constraints if there’s a separation oracle

Ellipsoid Algorithm in d dimensions

- Start with a big ellipsoid containing the feasible region
- Check each constraint to see if ellipsoid center is feasible
- If so, done
- Else find a violated constraint cutting the ellipsoid in half
- In poly(d) time find a new ellipsoid containing the half of the old ellipsoid containing the feasible region

Volume Argument

- Volume of new ellipsoid at most (1-1/d)*volume of old ellipsoid

- After d iterations, what is volume of new ellipsoid?
- After d^2L iterations, what is volume of new ellipsoid?

- Starting volume is $2^{\Theta(Ld)}$
  - Use Cramer's rule

- End volume is $2^{-\Theta(Ld\ \ )}$
  - Add a tiny amount to right hand side of each inequality $A_i \cdot x \leq b_i$
  - Feasible region could be a point, but after adding this, it has positive volume
  - If infeasible, then because of bit complexity L, after adding this, still infeasible

Time Complexity

- $\text{poly}(\text{d})$ iterations, in each just walk through m constraints to find a violated one
- Find description of new ellipsoid in poly(dL) time
  - Do some linear algebra
- Overall poly(mdL) time