15-451 Algorithms, Spring 2017
Recitation #8 Worksheet

A. Rock-Paper-Scissors with a Twist Suppose we have a non-standard game of rock-paper-scissors, which is still zero-sum, but with the following payoffs for the row player (Alice):

Bob plays

Alice plays		r	p	s
	r	0	-1	2
	p	1	0.5	-1
	s	-1	2	-1

1. If Alice decides to play $\mathbf{p} = (p_1, p_2, 1 - p_1 - p_2)$ as her strategy, what should Bob play to minimize the payoff to Alice? What is Alice's payoff if he does this.

2. Hence, write down the linear program that Alice must solve in order to find her best strategy $\mathbf{p}^\star$. You don't have to solve for the value of this game.

B. Taking Duals I. Consider this maximization linear program:

$$\begin{aligned} &\max(x_1 + 3x_2 - 2x_3) \\ \text{s.t.} \quad & x_1 + x_2 + 2x_3 \le 2 \\ & 7x_1 + 2x_2 + 5x_3 \le 6 \\ & 2x_1 + x_2 - x_3 \le 1 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

1. Write down its dual LP. (Is it a maximization or minimization problem? What are the variables? Constraints?)

2. Now write down the dual of this dual LP. Remember, since the dual will be a minimization LP, this dual's-dual will give a best lower bound on the dual.

Taking Duals II. Write down the dual of this minimization LP: *(Be careful, some inequalities are greater-than, some are less-than. And not all constraints have all variables.)*

$$\begin{aligned} &\max(x_1 - 3x_2 + 2x_3) \\ \text{s.t.} \quad & 3x_1 + 2x_3 \ge 2 \\ & 2x_2 - x_3 \le 5 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

C. Minimax from Duality (by Example). Let the row-player's payoffs be given by this (non-negative) matrix

	L	R
L	1	5
R	3	2

1. If the probabilities on the two rows are p_1 and p_2, write down an LP for the row player's optimal strategy:

2. Now take the dual of this LP. Show this dual is an LP computing the column player's optimal strategy. (And hence strong duality implies the minimax theorem.)

D. Coding up shortest paths as an LP: In lecture you saw that you can code up the s-t shortest-path problem as an LP. The input is a directed graph G with edge weights $w(e) \geq 0$, start node s, and a target t. We want to find a path from s to t of least weight.

1. *What are the variables?* Suppose we have a variable d_v for every vertex v, representing its distance from s. What are the constraints you should write?

2. *The objective?* We claim it is *maximize* d_t. Why does this make sense?

D'. Coding up shortest paths as an LP (Approach II): Have a variable x_e for each edge e, with constraints that $0 \leq x_e \leq 1$. (Think of $x_e = 1$ meaning we use that edge, and $x_e = 0$ meaning we don't, but of course the LP might assign fractional values.) Our goal is to minimize $\sum_e w(e)x_e$, subject to:

- One unit of "flow" leaves s: $\sum_{e=(s,v)} x_{sv} = 1$
- One unit of "flow" enters t: $\sum_{e=(v,t)} x_{vt} = 1$.
- For all $v \notin \{s, t\}$, we have flow-in = flow-out: $\sum_{e=(u,v)} x_{uv} = \sum_{e=(v,u)} x_{vu}$.

This is a min-cost flow (send 1 unit of s-t-flow with least cost). You can put edge-capacity 1, but since we send only 1 unit of flow, the capacity constraints don't matter.

1. Solve the LP to get an optimal solution. If we get back an integer solution (i.e., if $x_e \in \{0, 1\}$ for each edge e) argue that the edges with $x_e = 1$ give a shortest s-t path.

2. Suppose the optimal LP solution returns a fractional flow. Argue that any flow-carrying path from s to t is a shortest path.

Now, what if the LP solver returns fractional values? The claim is that in that case, you can get at least as good an LP solution by rerouting that flow on the shortest path. So there exists an optimal integer solution. (And a vertex solution, that simplex will find, will be integral.) We got lucky here that the fractional values didn't hurt us. For other problems (e.g., the non-bipartite matching problem, or 3SAT and vertex-cover that we'll see in the coming lectures) you can't necessarily convert a fractional solution into an integer one.