

15-451, Spring 2017, Recitation 12

Competitive Analysis and Random Incremental Analysis

1 Where to put the Safe?

An office building has one safe where valuables are kept. There are two rooms in the building numbered 1, 2. The distance between the rooms is 1. There is a sequence of requests where an employee in some room needs to access the safe. If the employee is in the room with the safe, the cost is 0. If the employee is in the other room, the cost is 1. Management is monitoring these activities, and has the option to move the safe from time to time to a different room. The cost of moving the safe is p .

The requests are adversarially generated, and future requests are unknown by management. After each request management has the option of moving the safe from one place to another. Management's goal is to obtain a deterministic algorithm with low total cost. (The total cost includes the employee movement costs plus the costs incurred by moving the safe.) The criterion of any management algorithm is the competitiveness of the algorithm, as defined in class.

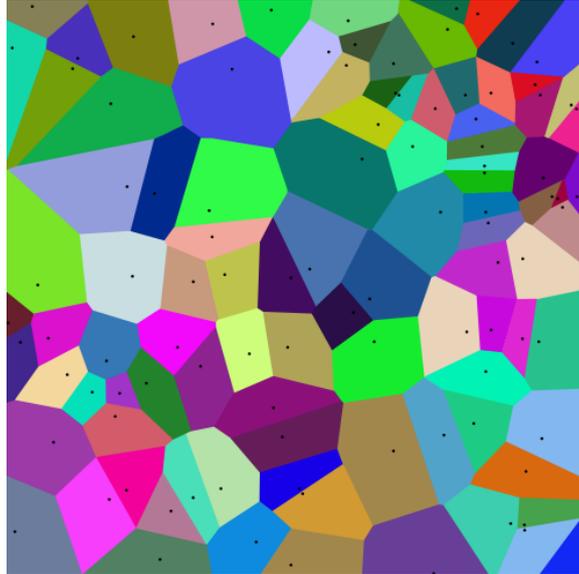
We will analyze a counter-based algorithm, where the threshold is $2p$. This means that when a request is from the room without the safe, the algorithm processes it in the current room (at a cost of 1), and increments the counter. If the counter has reached $2p$ it moves the safe to the just requested room (at a cost of p), and resets the counter to 0. Call this on-line algorithm A , and call the adversary's algorithm B .

Prove that this algorithm is 3-competitive. Hint: use the following potential function. Here S_X is where algorithm X currently has the safe located.

$$\Phi(S_A, c, S_B) = \begin{cases} 2c & \text{if } S_A = S_B \\ 3p - c & \text{if } S_A \neq S_B \end{cases}$$

2 Painting

A sequence of n distinct points p_1, p_2, \dots, p_n in the unit square is specified. In a *painted voronoi diagram* the part of the unit square closest to point p_i is painted with color c_i . Here is an example:



This problem concerns the amount of paint used when by a certain painting algorithm:

Painting Algorithm: Insert the points one at a time in the given order. Each time a point is inserted, paint the region of the unit square that is closer to point p_i than any other point that has been inserted so far.

Note that one unit of paint is required to paint the entire square.

1. Devise a sequence of n points and prove that for this sequence the painting algorithm uses $\Omega(n)$ paint.

2. Returning to the original formulation, where the sequence of points is specified, suppose that before applying the painting algorithm, we randomly permute the n points. Prove that the expected total amount of paint used by this algorithm is at most $1 + \ln n$. Note that this result is independent of the set of points.