3. **Bird Walk**

There is a huge circle of wire hanging in the air above the ground. The circumference of the circle is $D$. There are $n$ pairs of birds. Each of these $n$ pairs has formed a nest at an arbitrary (not random) location on this circle. See Figure 1a:

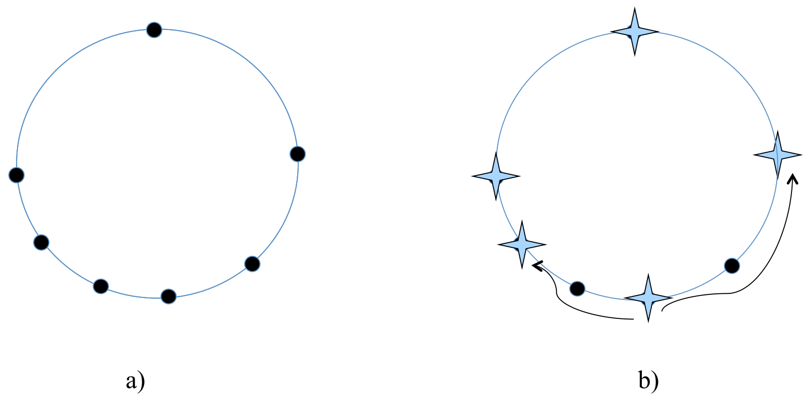

Figure 1: a) The original location of nests; b) The new nest is at the bottom and birds travel in either directions until reaching the nearest nests.

On some stormy night, all pairs of birds flew away to a nearby forest. However, each pair was still together even in such a moment of despair. Unfortunately, all the nests fell down that night. In subsequent days the birds start returning to the circle to find the status of their nests. Every day one pair of birds returns to the location where their nest was located. The first pair of birds arrives, places a new nest at the location of their old nest, and leaves in search of food. Each subsequent pair of birds in some order does the following: On returning to the earlier location of their nest, they place a new nest at the same location. Then they start walking in opposite directions along the circle until each of the two birds finds a nest (see Figure 3b). After this they fly off looking for food. We are interested in the total distance traveled by all the birds along the circle.

(10) (a) Give locations of the nests along with an ordering of the birds' returns for which the total distanced traveled by the birds is $\Omega(D\,n)$.

**Solution:** Place the nest at equal spacings around the circle and have the birds return in CCW order of their nest.

The $i$th return pair will walk a distance of $D(n-i)/n$. Giving $\sum_{i=1}^{n}(D/n)i = \omega(Dn)$.

(20) (b) Show that if the pairs of birds return in random order, then the total expected distance walked by all the birds is $O(D \log n)$ independent of the location of the original nests on the circle. Hint: Use backward analysis!

**Solution:** The expected cost to adding back a nest is $2D/(n-i)$, where $i = 0, \ldots, n-1$. The total cost

$$\sum_{i=0}^{n-1} \frac{2D}{n-i} = \sum_{i=1}^{n} \frac{2D}{i} = 2DH_n = \Theta(D \log n)$$