# Experts and Multiplicative Weights

slides from Avrim Blum

## Using "expert" advice

Say we want to predict the stock market.

- We solicit n "experts" for their advice. (Will the market go up or down?)
- We then want to use their advice somehow to make our prediction. E.g.,

| Expt 1 | Expt 2 | Expt 3 | neighbor's dog | truth |
|---|---|---|---|---|
| down | up | up | up | up |
| down | up | up | down | down |
| … | … | … | … | … |

Basic question: Is there a strategy that allows us to do nearly as well as best of these in hindsight?

["expert" = someone with an opinion. Not necessarily someone who knows anything.]

## Simpler question

- We have n "experts".
- One of these is perfect (never makes a mistake). We just don't know which one.
- Can we find a strategy that makes no more than lg(n) mistakes?

Answer: sure. Just take majority vote over all experts that have been correct so far.

- Each mistake cuts # available by factor of 2.
- Note: this means ok for n to be very large.

## What if no expert is perfect?

Intuition:

Making a mistake doesn't completely disqualify an expert. So, instead of crossing off, just lower its weight.

Weighted Majority Alg:

- Start with all experts having weight 1.
- Predict based on weighted majority vote.
- Penalize mistakes by cutting weight in half.

| | | | | | prediction | correct |
|---|---|---|---|---|---|---|
| weights | 1 | 1 | 1 | 1 | | |
| predictions | Y | Y | Y | N | Y | Y |
| weights | 1 | 1 | 1 | .5 | | |
| predictions | Y | N | N | Y | N | Y |
| weights | 1 | .5 | .5 | .5 | | |

## Analysis: do nearly as well as best expert in hindsight

- M = # mistakes we've made so far.
- m = # mistakes best expert has made so far.
- W = total weight (starts at n).
- After each mistake, W drops by at least 25%.

  So, after M mistakes, W is at most $n(3/4)^M$.
- Weight of best expert is $(1/2)^m$. So,

$$(1/2)^m \leq n(3/4)^M$$
$$(4/3)^M \leq n2^m$$
$$M \leq 2.4(m + \lg n)$$

So, if m is small, then M is pretty small too.

## Randomized Weighted Majority

Instead of taking majority vote, use weights as probabilities. (e.g., if 70% on up, 30% on down, then pick 70:30)

Idea: smooth out the worst case.

Also, generalize 1/2 to 1- $\epsilon$.

Solves to

$$M \leq \frac{-m \ln(1-\epsilon) + \ln n}{\epsilon} \approx (1+\epsilon/2)m + \frac{\ln n}{\epsilon}$$

M = expected #mistakes

$M \leq 1.39\,m + 2 \ln n$ (when $\epsilon = 1/2$)

$M \leq 1.15\,m + 4 \ln n$ (when $\epsilon = 1/4$)

$M \leq 1.07\,m + 8 \ln n$ (when $\epsilon = 1/8$)

## Analysis

Say at time t we have fraction $F_t$ of weight on experts that made mistake

So we have probability $F_t$ of making a mistake, and we remove $\epsilon F_t$ fraction of total weight

$W_{final} = n(1 - \epsilon F_1)(1 - \epsilon F_2) \ldots$

$\ln W_{final} = \ln n + \sum_t (1 - \epsilon F_t) \leq \ln n - \epsilon \sum_t F_t$

(using $\ln(1 - x) \leq -x$ )

But $\sum_t F_t$ = expected number of mistakes = $\epsilon M$.

If best expert makes m mistakes then $\ln(W_{final}) \geq \ln(1 - \epsilon)^m$

$= m \ln(1 - \epsilon)$

Now solve $\ln n - \epsilon M \geq m \ln(1 - \epsilon)$.

$$M \leq \frac{-m \ln(1 - \epsilon) + \ln n}{\epsilon} \approx (1 + \epsilon/2)m + \frac{\ln n}{\epsilon}$$

## An application

Can use this for repeated play of matrix game

Consider cost matrix where all entries are 0 or 1

Rows are different experts. Start each with weight 1.

Notice that RWM is equivalent to "pick expert i with probability $(w_i / \sum_j w_j)$ and go with it"

Can apply with experts are actions rather than predictions

$F_t$ = fraction of weight on rows that had "1" in adversary's column.

Analysis shows that we can do nearly as well as best expert in hindsight.

## An application

In fact, algorithm/analysis extends to costs in [0,1] not just in {0,1}

We assign weights $w_i$, inducing probabilities $p_i = (w_i / \sum_j w_j)$

We choose a random row according to this distribution p.

Adversary chooses column. This gives column vector c.

We pay expected cost $p.c = \sum_i p_i c_i$.

Update: $w_i = w_i (1 - \epsilon c_i)$

## RWM: matrix view

$(1-\varepsilon c_1^2)$ $(1-\varepsilon c_1^1)$ 1

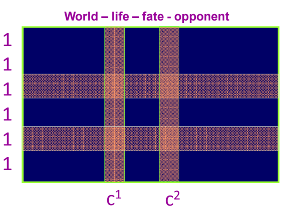
$(1-\varepsilon c_2^2)$ $(1-\varepsilon c_2^1)$ 1

$(1-\varepsilon c_3^2)$ $(1-\varepsilon c_3^1)$ 1

. . 1

. . 1

$(1-\varepsilon c_n^2)$ $(1-\varepsilon c_n^1)$ 1




scaling so costs in [0,1]

$$E[cost] \leq (1 + \epsilon/2)OPT + \frac{\ln n}{\epsilon} \leq OPT + \epsilon T/2 + \frac{\ln n}{\epsilon}$$

Since OPT over T steps is at most T

## A proof of the Minimax Theorem

RWM gives a clean simple proof of the minimax theorem.

Suppose for contradiction minimax theorem was false.

This means some game G has $V_C > V_R$

If Column player commits first, there exists a row that gets the Row player at least $V_C$

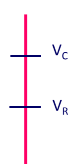
But if Row player has to commit first, the Column player can make him get only $V_R$

Scale matrix so payoffs to row are in [-1,0].

Observe: payoffs of –P to row = cost of P to row

⇒ can view as costs and hence use RWM

Also, say $V_R = V_C - \delta$




## A proof of the Minimax Theorem (contd)

Now consider RWM algorithm against column who plays optimally against row's distribution (at each time).

In T steps,

1) Alg gets ≥ [best row in hindsight] - $\epsilon T/2$ - $(\log n)/\epsilon$

[by guarantee of the RWM algorithm]

2) best row in hindsight ≥ $T^*V_C$

[if row player plays optimally against empirical distr. of column player]

3) But Alg ≤ $T^*V_R$

[since each time opponent knows your distribution]

By (2)-(3), gap between alg and best row is ≥ $\delta^*T$.

Contradicts (1) for $\epsilon = \delta/2$ once we have $T \geq (\ln n)/\epsilon^2$.