15-414 — Bug Catching — Fall 2006

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Assignment 1 Solutions

1 Tautologies

For each of the following formula report if it is a tautology or give an assignment which falsifies the formula.

[Hint: Try to solve these without constructing truth tables. Think about falsifying the formula.]

(a) $((q \rightarrow (p \vee (q \rightarrow p))) \vee \neg(p \rightarrow q)) \rightarrow p$

Formula is falsified when both p, q are false.

(b) $((a \rightarrow x) \wedge (a \Leftrightarrow \neg y) \wedge (y \Leftrightarrow (z \wedge w))) \rightarrow ((x \Leftrightarrow y) \rightarrow (z \rightarrow w))$

Tautology

(c) $((p \rightarrow q) \wedge (q \Leftrightarrow T)) \rightarrow (p \Leftrightarrow T)$, where T denotes *true*.

Formula is falsified when p is false and q is true

2 Conversions to Normal forms without introducing new variables

(a) Convert following formula to CNF: $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$
$\Leftrightarrow (\neg(p \rightarrow (q \rightarrow r))) \vee ((p \rightarrow q) \rightarrow (p \rightarrow r))$
$\Leftrightarrow (\neg(p \rightarrow (q \rightarrow r))) \vee ((p \rightarrow q) \rightarrow (\neg p \vee r))$
$\Leftrightarrow (p \wedge \neg(q \rightarrow r)) \vee ((p \rightarrow q) \rightarrow (\neg p \vee r))$
$\Leftrightarrow (p \wedge (q \wedge \neg r)) \vee ((p \rightarrow q) \rightarrow (\neg p \vee r))$
$\Leftrightarrow (p \wedge q \wedge \neg r) \vee (\neg(p \rightarrow q) \vee (\neg p \vee r))$
$\Leftrightarrow (p \wedge q \wedge \neg r) \vee ((p \wedge \neg q) \vee \neg p \vee r)$
$\Leftrightarrow (p \wedge q \wedge \neg r) \vee (\neg q \vee \neg p \vee r)$
$\Leftrightarrow (p \wedge q \wedge \neg r) \vee \neg(q \wedge p \wedge \neg r)$
$\Leftrightarrow T$
The CNF formula is just true.

(b) Convert following formula to CNF: $\neg(p \rightarrow (\neg(q \wedge (\neg p \rightarrow q))))$
$\Leftrightarrow (p \wedge \neg(\neg(q \wedge (\neg p \rightarrow q))))$
$\Leftrightarrow (p \wedge (q \wedge (\neg p \rightarrow q)))$
$\Leftrightarrow (p \wedge (q \wedge (p \vee q)))$
$\Leftrightarrow p \wedge q \wedge (p \vee q)$

(c) Convert following formula to DNF: $(x \vee y) \wedge (y \rightarrow \neg x) \wedge (z \Leftrightarrow \neg x)$
$\Leftrightarrow (x \vee y) \wedge (\neg y \vee \neg x) \wedge (z \rightarrow \neg x) \wedge (\neg x \rightarrow z)$
$\Leftrightarrow (x \vee y) \wedge (\neg y \vee \neg x) \wedge (\neg z \vee \neg x) \wedge (x \vee z)$

Just multiplying out clauses and removing terms containing conjunction of negated literals, we get
$\Leftrightarrow (x \wedge \neg y \wedge \neg z) \vee (y \wedge \neg x \wedge z)$

3 Convert to CNF by introducing new variables

(a) $(a \wedge b) \vee (c \wedge d) \vee (e \wedge f) \vee (g \wedge h)$

We introduce a new variables x, y, z, w for $(a \wedge b), (c \wedge d), (e \wedge f), (g \wedge h)$, respectively. Thus, the originial formula can be written as following equi-satisfiable formula
$(x \vee y \vee \vee z \vee w) \wedge (x \Leftrightarrow a \wedge b) \wedge (y \Leftrightarrow c \wedge d) \wedge (z \Leftrightarrow e \wedge f) \wedge (w \Leftrightarrow g \wedge h)$
It can be easily converted to following the CNF formula
$(x \vee y \vee \vee z \vee w) \wedge$
$(\neg x \vee a) \wedge (\neg x \vee b) \wedge (\neg a \vee \neg b \vee x)$
$(\neg y \vee c) \wedge (\neg y \vee d) \wedge (\neg c \vee \neg d \vee y)$
$(\neg z \vee e) \wedge (\neg x \vee f) \wedge (\neg e \vee \neg f \vee z)$
$(\neg w \vee g) \wedge (\neg w \vee h) \wedge (\neg g \vee \neg h \vee w)$

(b) $(x_1 \rightarrow (x_2 \rightarrow x_3)) \vee (x_4 \rightarrow x_5)$
This formula does not need any new variables. It is written in CNF form in a straightforward manner.
$(\neg x_1 \vee \neg x_2 \vee x_3 \vee \neg x_4 \vee x_5)$

4 Applying Davis-Putnam method

Check whether the following formulas are satisfiable by applying the Davis-Putnam rules. Also report a satisfying assignment when a formula is satisfiable.

(a) $(a \vee \neg b) \wedge (d \vee \neg e) \wedge (b \vee \neg c) \wedge (c \vee \neg d) \wedge (\neg a \vee z) \wedge e \wedge \neg z$

UNSATIASFIABLE

(b) $(\neg a \vee \neg b) \wedge (\neg a \vee \neg c) \wedge (\neg b \vee \neg c) \wedge (\neg d \vee \neg e) \wedge (\neg d \vee \neg f) \wedge (\neg e \vee \neg f) \wedge (a \vee d) \wedge (b \vee e) \wedge (c \vee f)$

UNSATISFIABLE

5 Equivalence

(a) Let $\mathcal{V}_\phi(A)$ denote the value of a propositional formula A under an assignment ϕ. Show that if A is a well formed formula (wff) in which no propositional connective other than $\Leftrightarrow$ occurs, and ϕ is an assignment, then $\mathcal{V}_\phi(A) = F$ iff the number of occurrences of variables p in A such that $\phi(p) = F$ is odd. Note that F denotes $false$.

[Hint: Use induction on the structure of the formula (to be explained in class).]

(b) Prove that a wff of propositional calculus containing only the connective $\Leftrightarrow$ is a tautology iff each propositional variable occurs an even number of times.

[Hint: Use the part (a).]

6 Pigeonhole formulas

The *pigeonhole principle* states that if n pigeons are put into m pigeonholes, and if $n > m$, then at least one pigeonhole must contain more than one pigeon.

Given n pigeons and m pigeonholes, we wish to write a CNF formula G such that G is satisfiable iff each pigeon can be put in some pigeonhole and each pigeonhole has atmost one pigeon.

Let $x_{i,j}$ denote a propositional variable which is true when pigeonhole i contains pigeon j. We will give the formula for G when $n = 3$ and $m = 3$.

$E := (x_{1,1} \vee x_{2,1} \vee x_{3,1}) \wedge (x_{1,2} \vee x_{2,2} \vee x_{3,2}) \wedge (x_{1,3} \vee x_{2,3} \vee x_{3,3})$

$H_1 := (\neg x_{1,1} \vee \neg x_{1,2}) \wedge (\neg x_{1,1} \vee \neg x_{1,3}) \wedge (\neg x_{1,2} \vee \neg x_{1,3})$

$H_2 := (\neg x_{2,1} \vee \neg x_{2,2}) \wedge (\neg x_{2,1} \vee \neg x_{2,3}) \wedge (\neg x_{2,2} \vee \neg x_{2,3})$

$H_3 := (\neg x_{3,1} \vee \neg x_{3,2}) \wedge (\neg x_{3,1} \vee \neg x_{3,3}) \wedge (\neg x_{3,2} \vee \neg x_{3,3})$

$G := E \wedge H_1 \wedge H_2 \wedge H_3$

(a) What is the meaning of clause $(x_{1,3} \vee x_{2,3} \vee x_{3,3})$.

pigeon 3 is contained in at least one of pigeonholes 1,2,3

(b) What does E stand for?

E means pigeons 1,2, and 3 are each contained in at least one of pigeonholes 1,2,3

(c) Explain the meaning of H_1, H_2, H_3.

H_i **means that pigeonhole** i **cannot contain more than one pigeon**

(d) Is G satisfiable? If yes, give a satisfying assignment.

Yes. A satisfying assignment is $x_{i,i} = true$ **where** $1 \leq i \leq 3$ **and** $x_{i,j} = false$ **for** $i \neq j, 1 \leq i, j \leq 3$**.**

(e) Write G when $n = 3$ and $m = 2$. Is G satisfiable now?

$E := (x_{1,1} \vee x_{2,1}) \wedge (x_{1,2} \vee x_{2,2}) \wedge (x_{1,3} \vee x_{2,3})$

$H_1 := (\neg x_{1,1} \vee \neg x_{1,2}) \wedge (\neg x_{1,1} \vee \neg x_{1,3}) \wedge (\neg x_{1,2} \vee \neg x_{1,3})$

$H_2 := (\neg x_{2,1} \vee \neg x_{2,2}) \wedge (\neg x_{2,1} \vee \neg x_{2,3}) \wedge (\neg x_{2,2} \vee \neg x_{2,3})$

$G := E \wedge H_1 \wedge H_2$

G **is not satisfiable.**