Great Theoretical Ideas In Computer Science

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CS 15-251 Fall 2006

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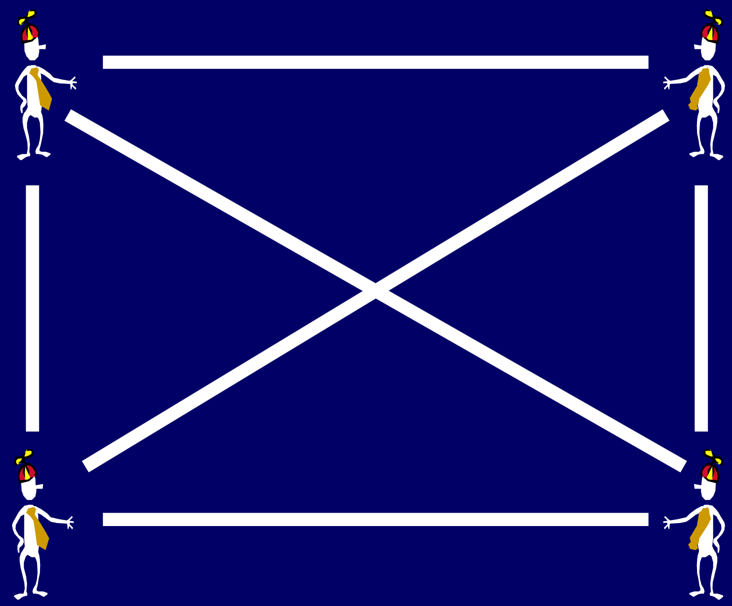
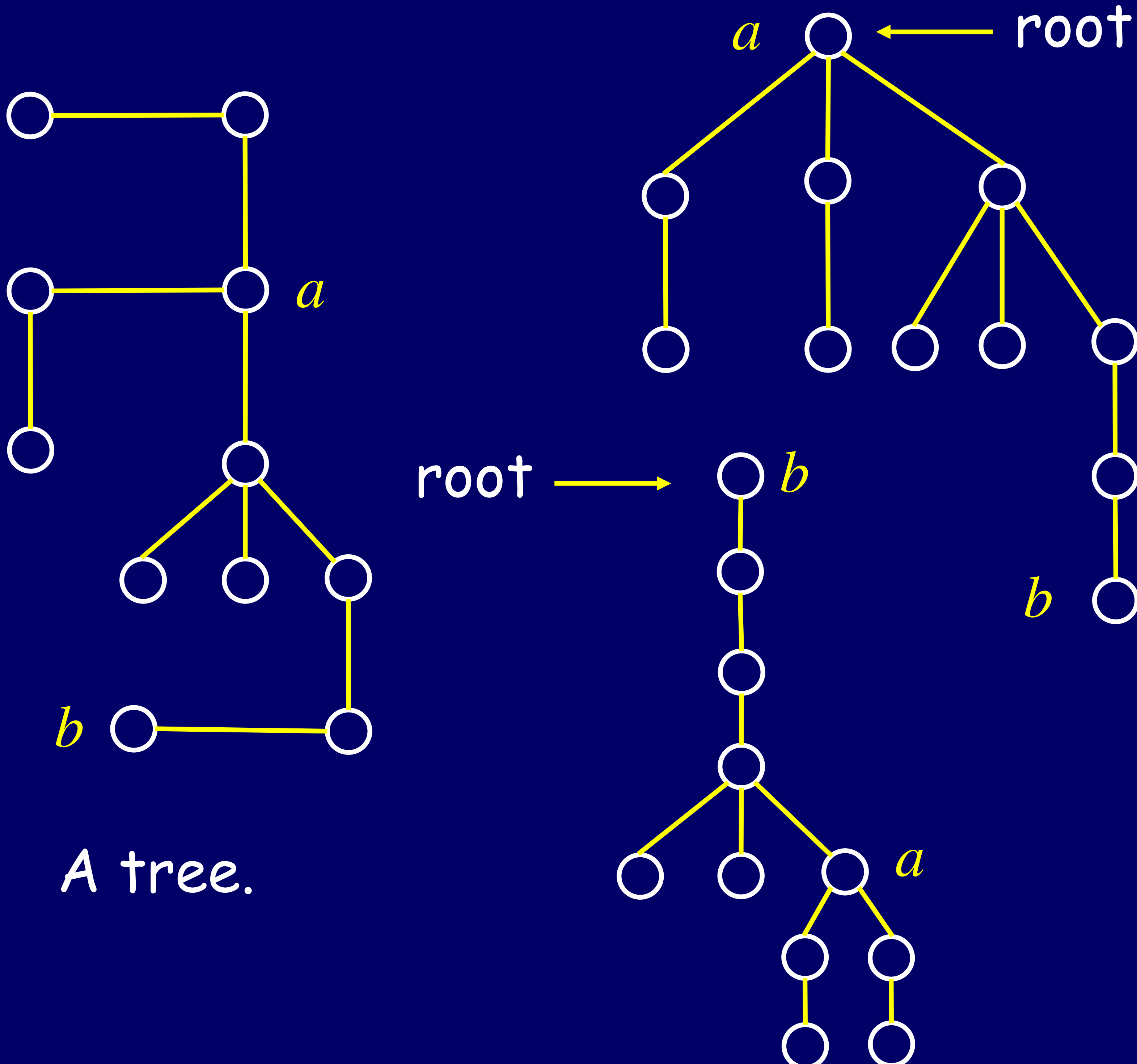
Carnegie Mellon University

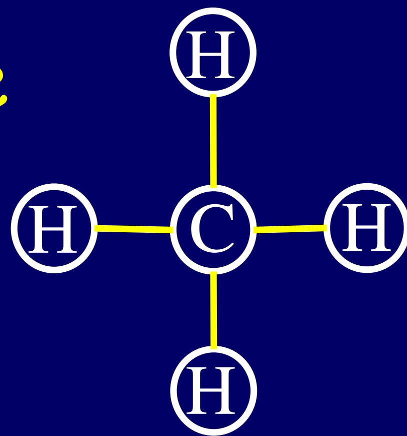
Graphs



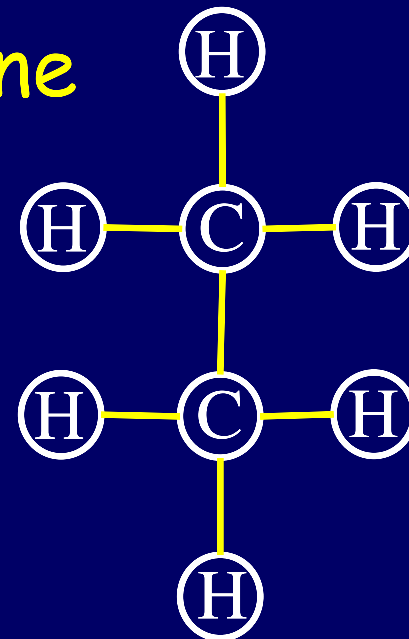


A tree.

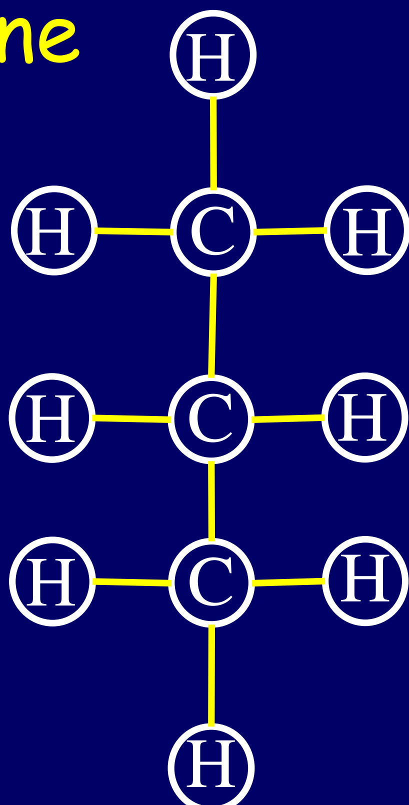
methane



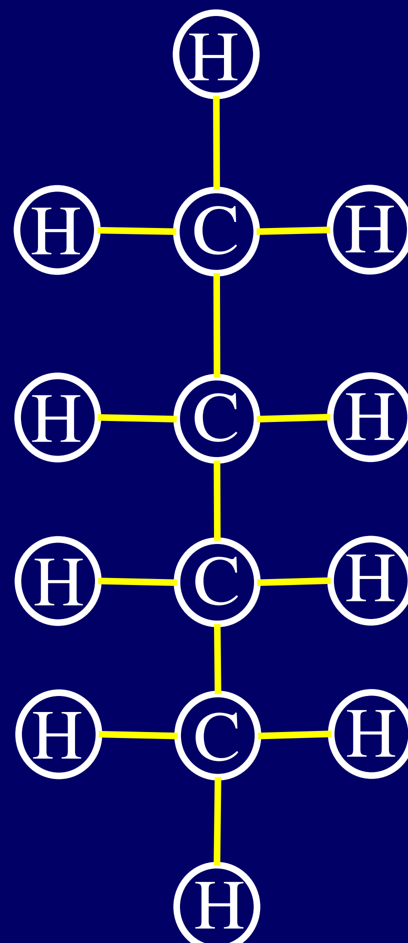
ethane



propane

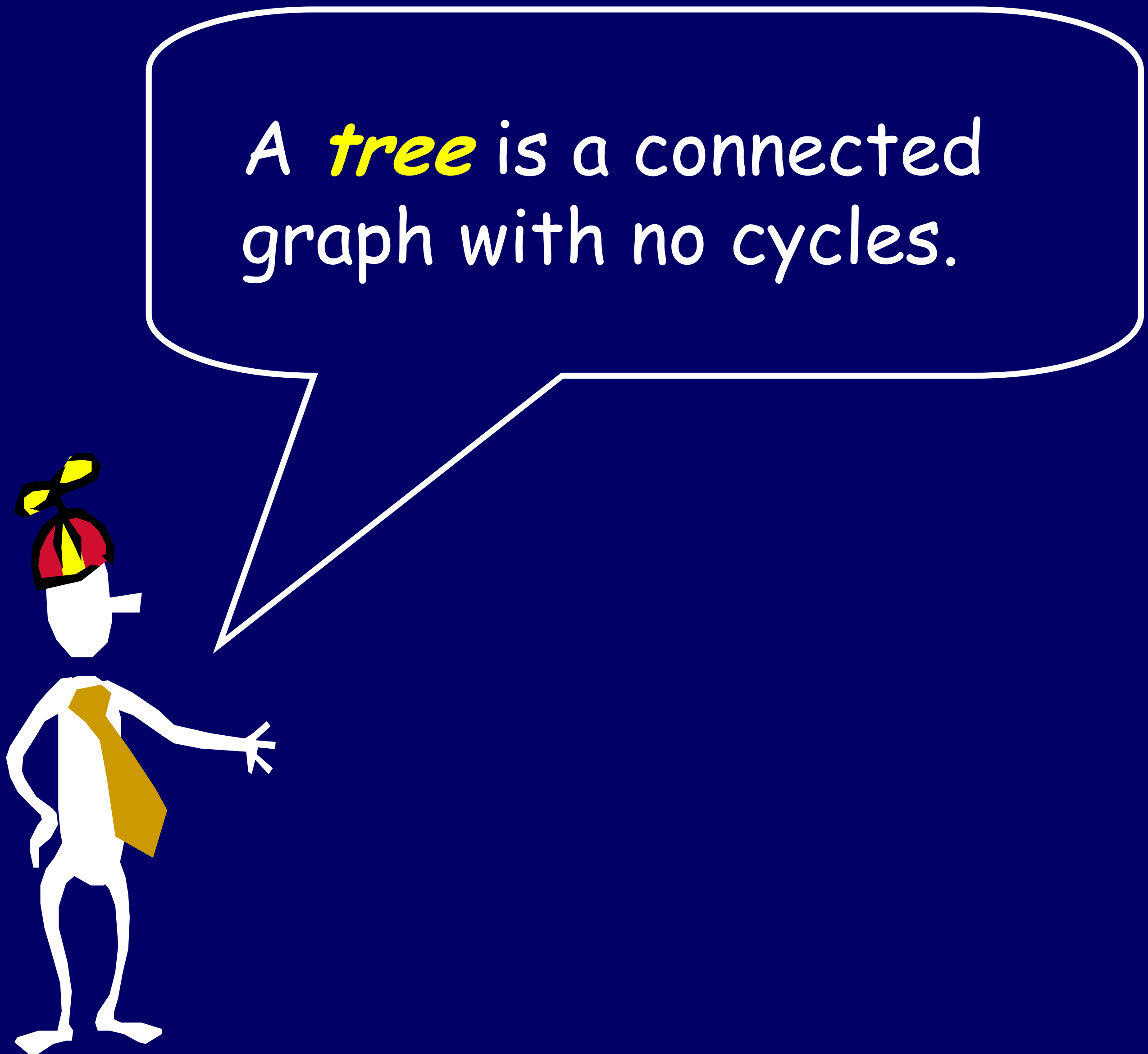


butane

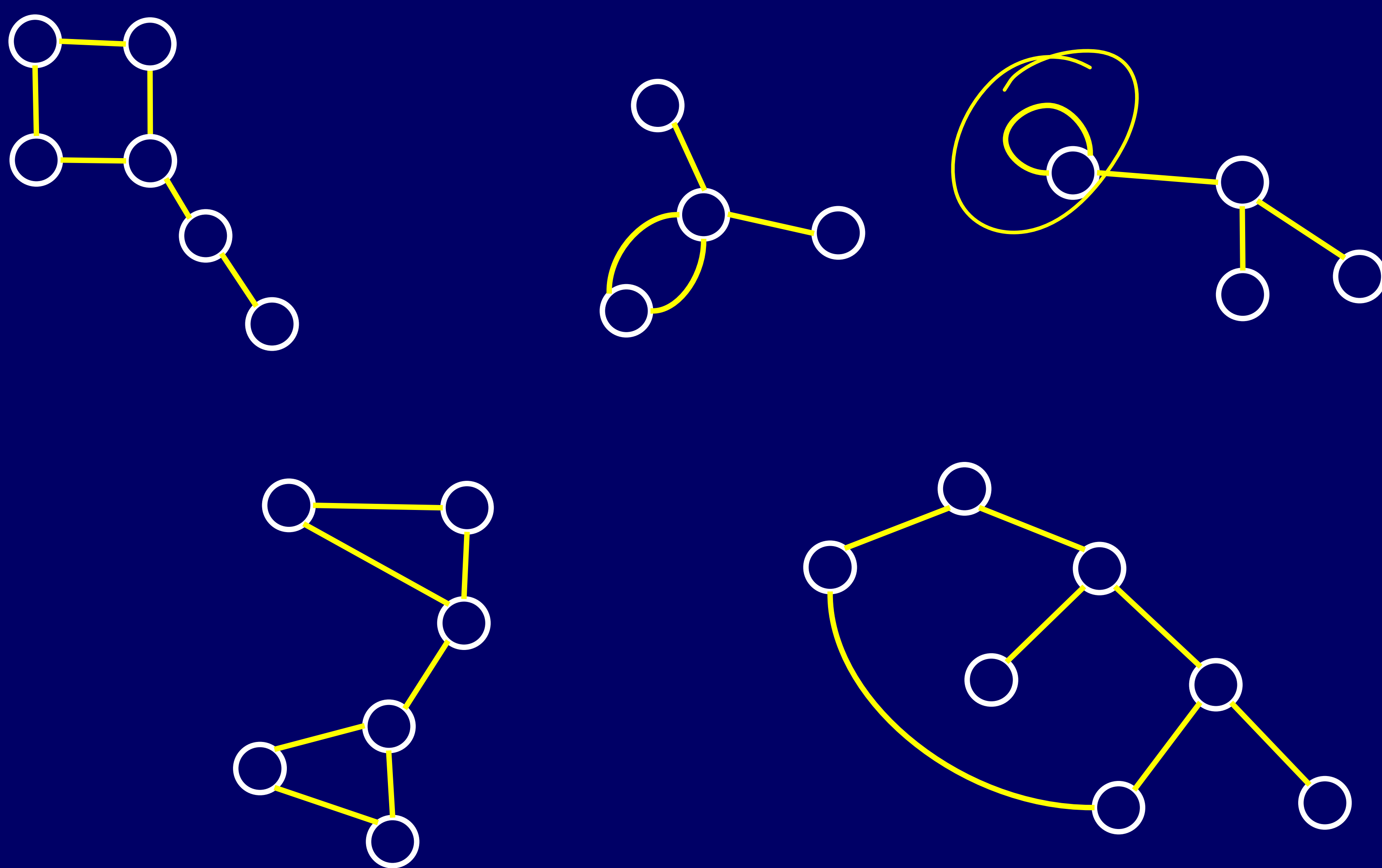


some saturated hydrocarbons

Putting a picture into words…



These are not trees...

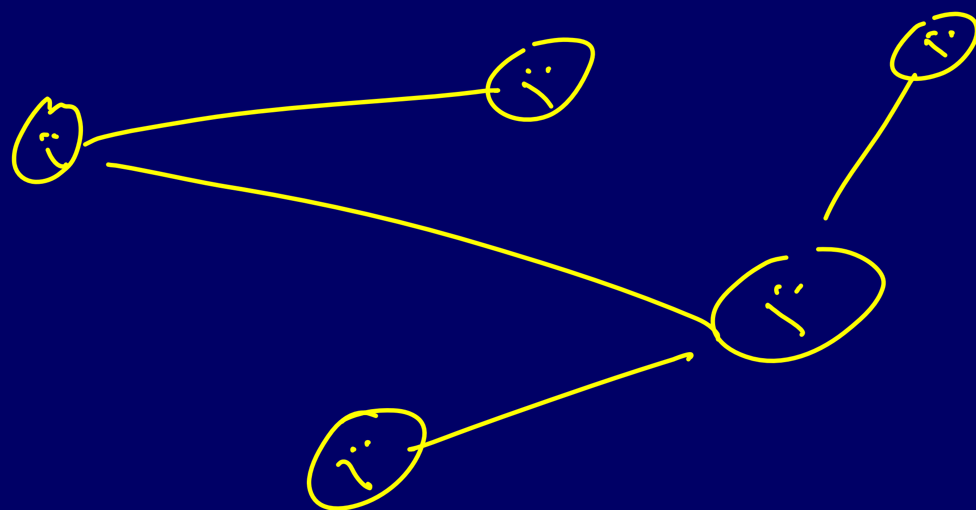


The Shy People Party

At the shy people party, people enter one-by-one, and as a person comes in, (s)he shakes hand with only one person already at the party.

Prove that at a shy party with n people (n >= 2), at least two people have shaken hands with only one other person.

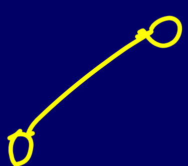
The Shy People Party



adding one person :

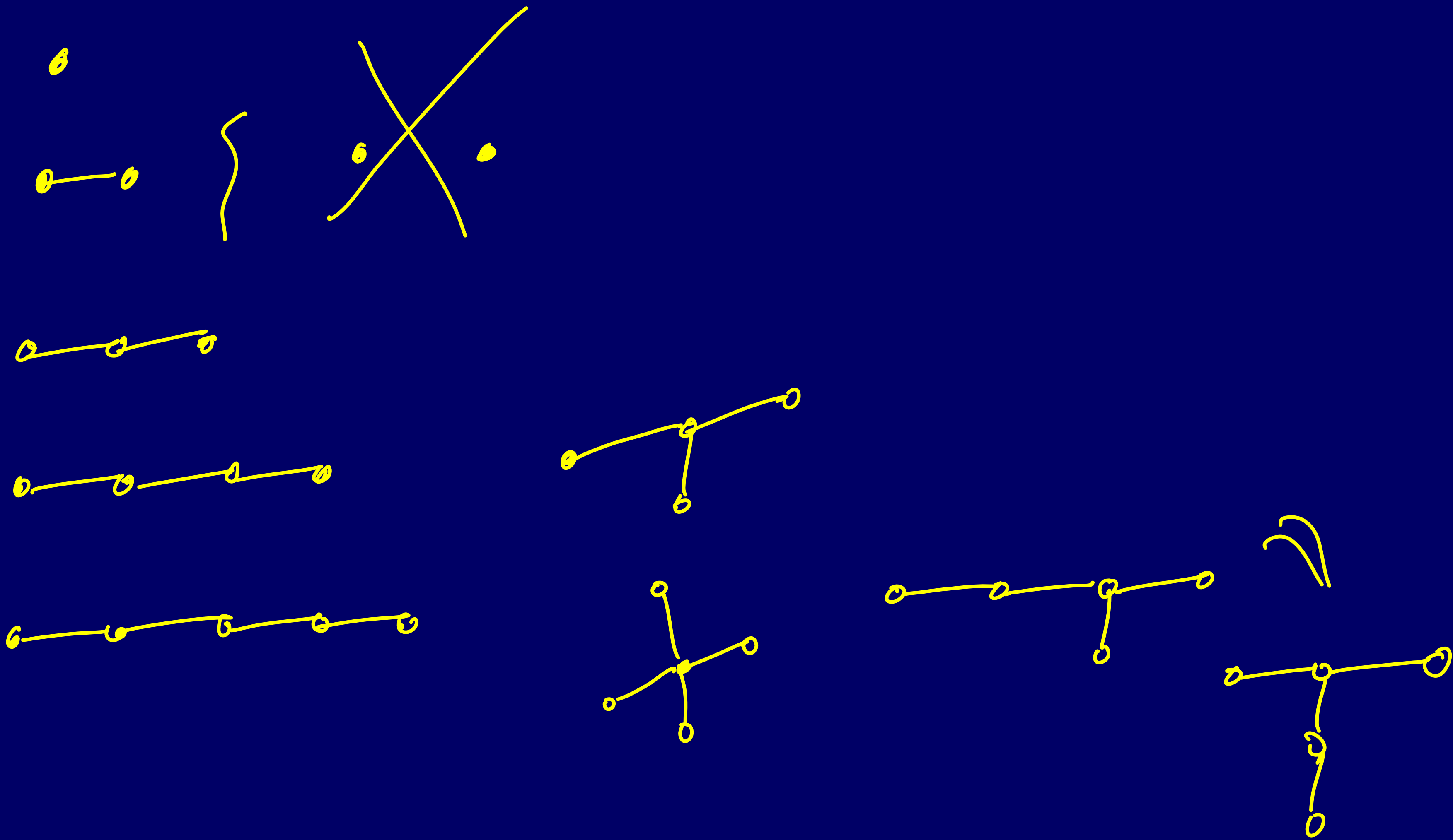
Case 1: shakes hands with a "1 shaker"

1 shakers stays same



Case 2: shakes hands with someone who has shaken > 1 hand.

How many trees on 1-6 vertices?



We'll pass around a piece of paper. Draw a new 8-node tree, and put your name next to it. (There are 23 of them…)

Theorem: Let G be a graph with n nodes and e edges.

The following are equivalent:

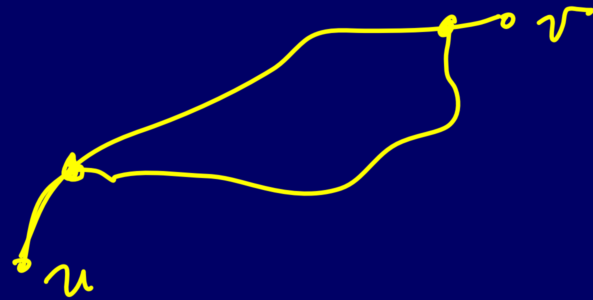
1. G is a tree (connected, acyclic)
2. Every two nodes of G are joined by a unique path
3. G is connected and $n = e + 1$
4. G is acyclic and $n = e + 1$
5. G is acyclic and if any two nonadjacent points are joined by a line, the resulting graph has exactly one cycle.

To prove this, it suffices to show

$$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$$

if G is a tree $\Rightarrow$ every two nodes are joined by a unique path.

$1 \Rightarrow 2$



$\Rightarrow$ a cycle.

$2 \Rightarrow 3$ every pair of nodes connected by a unique path $\Rightarrow$

G is connected and $n = e+1$

remove an edge

$n_2 = e_2 + 1$

$n_1 = e_1 + 1$

$n = n_1 + n_2 = e_1 + e_2 + 2$

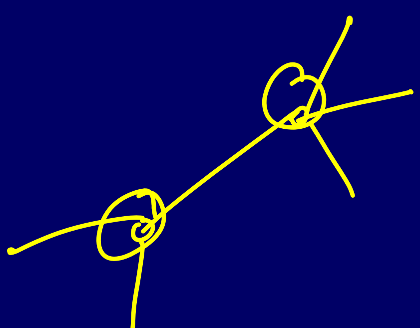
$= e + 1$

Corollary: Every nontrivial tree has at least two endpoints (points of degree 1)

$$n = n_1 + n_2 + n_3 + \ldots$$

n_i = # of nodes of degree i

$$n = e + 1$$

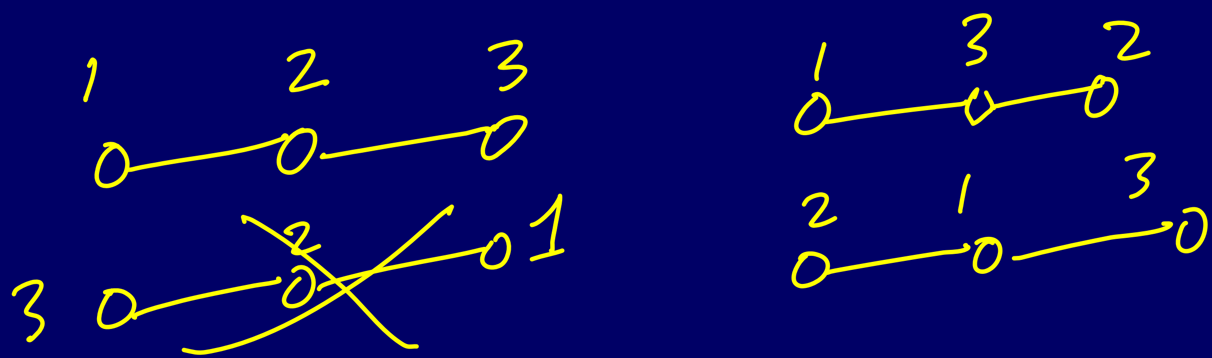


$$2e = 2n - 2$$

total degree

Question:

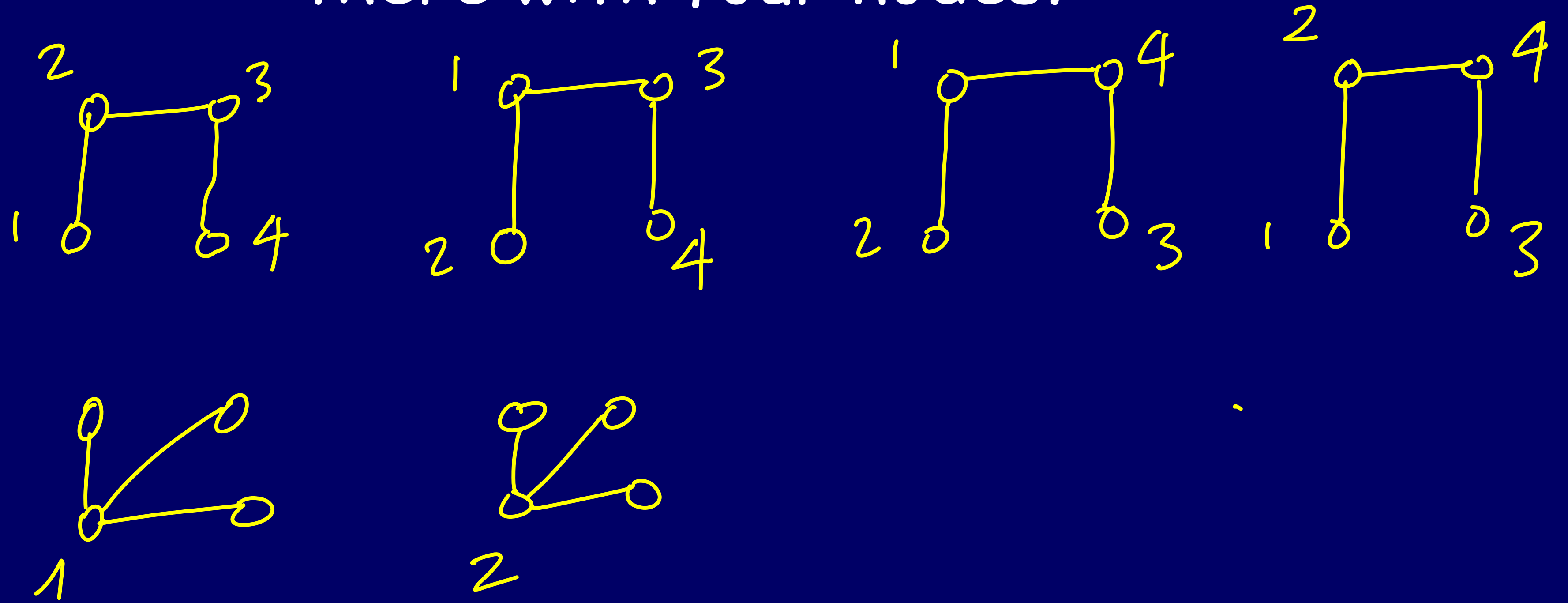
How many *labeled* trees are there with three nodes?



3 labelings.

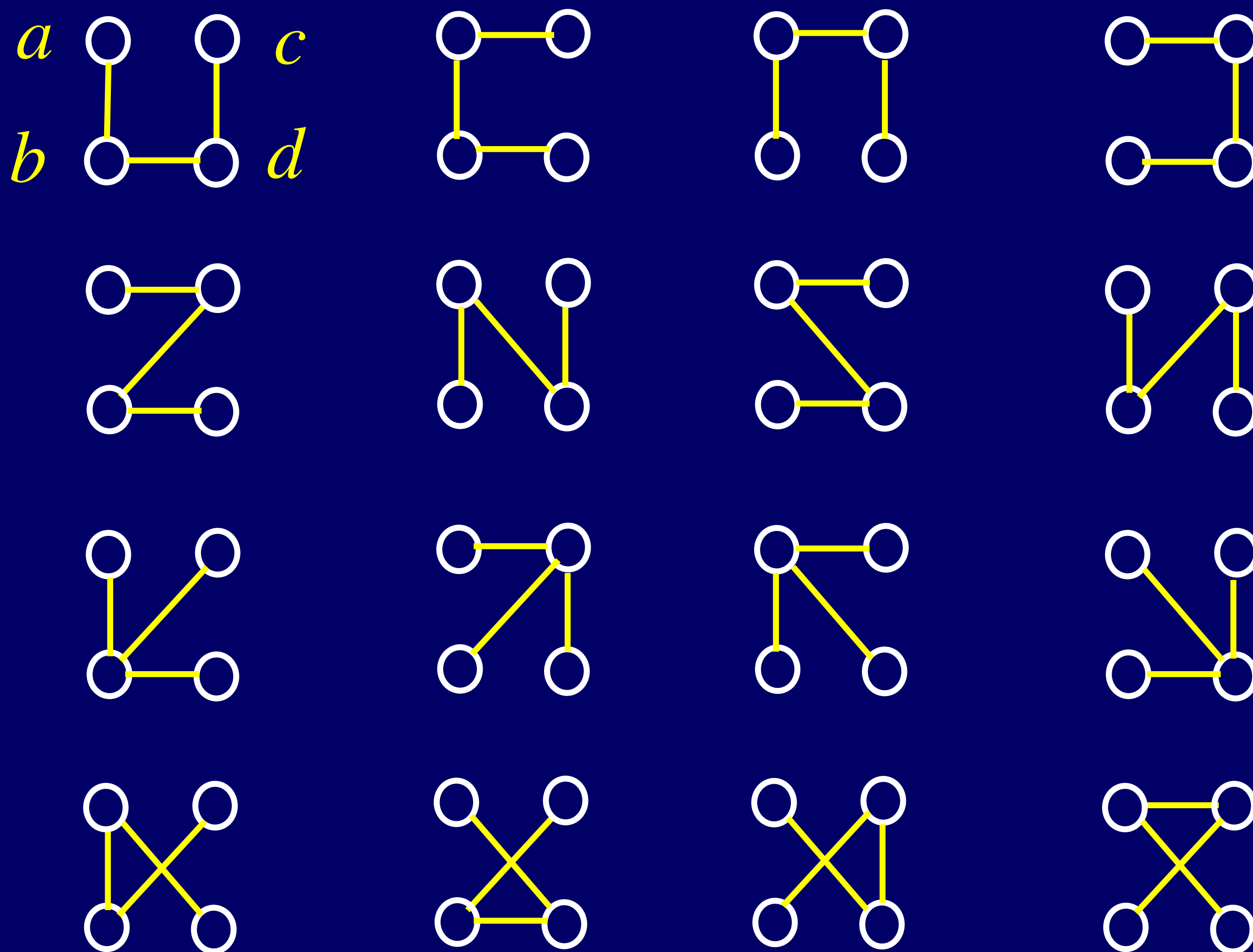
Question:

How many *labeled* trees are there with four nodes?



2 trees

16 labeled trees



Question:

How many labeled trees are there with five nodes?

$T_3 = 3$

$T_4 = 16$

$T_5 = 125$

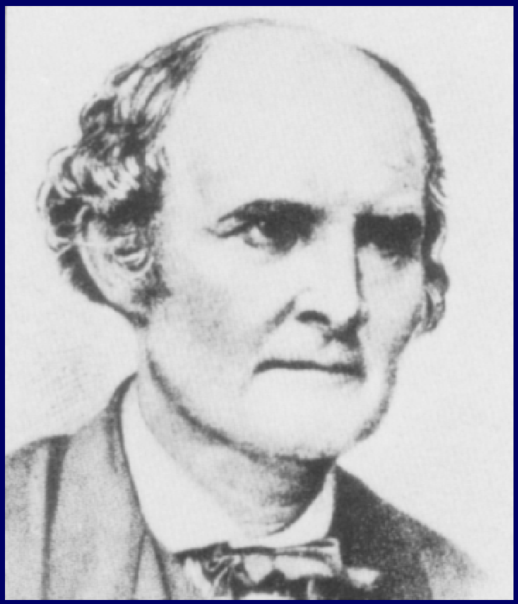
$T_n \stackrel{?}{=} n^{n-2}$

Question:

How many labeled trees on n nodes are there?

Cayley's formula

The number of labeled trees on n nodes is



$$\boxed{n^{n-2}}$$

The proof will use the correspondence principle.

Each labeled tree on n nodes

corresponds to

A sequence in $\{1,2,\ldots,n\}^{n-2}$ that is, (n-2) numbers, each in the range [1..n]

How to make a sequence from a tree.

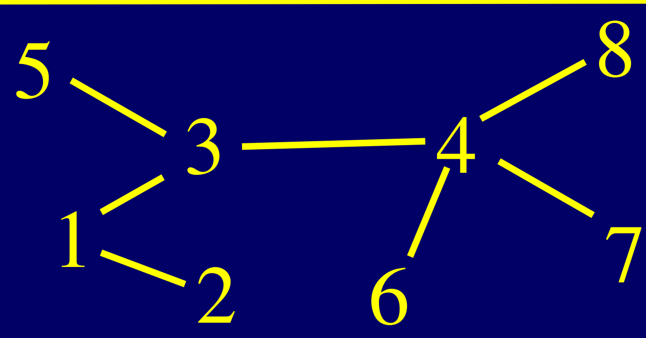
Loop through i from 1 to n-2

Let l be the degree-1 node with the lowest label.

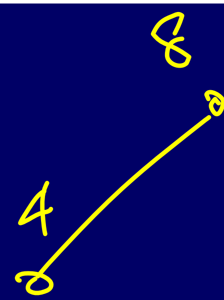
Define the i^{th} element of the sequence as the label of the node adjacent to l.

Delete the node l from the tree.

Example:



$$\langle 1,3,3,4,4,4 \rangle$$



$\langle 1, 3, 3, 4, 4, 4 \rangle$

How to reconstruct the unique tree from a sequence S.

Let $I = \{1, 2, 3, \ldots, n\}$

Loop until $S = \varepsilon$ ← empty sequence

Let l = smallest # in I but not in S

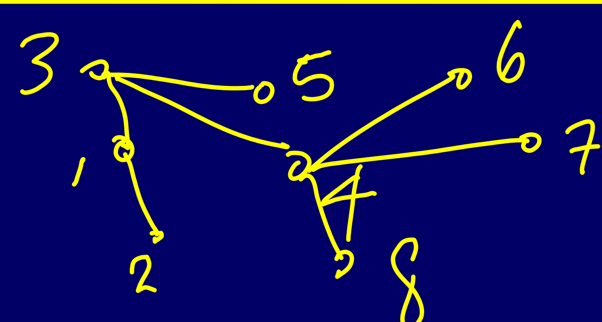
Let s = first label in sequence S

- Add edge $\{l, s\}$ to the tree.
- Delete l from I.
- Delete s from S.

Add edge $\{l, s\}$ to the tree, where $I = \{l, s\}$

$S = \langle \cancel{1}, \cancel{3}, \cancel{3}, \cancel{4}, \cancel{4}, \cancel{4} \rangle$

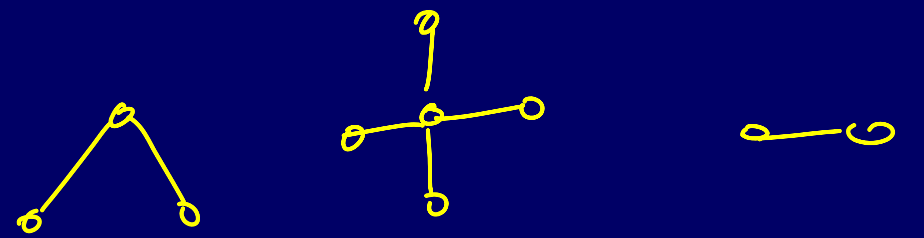
$I: \{\cancel{1}, \cancel{2}, \cancel{3}, \underbrace{4, \cancel{5}, \cancel{6}, \cancel{7}, 8}\}$



Another example

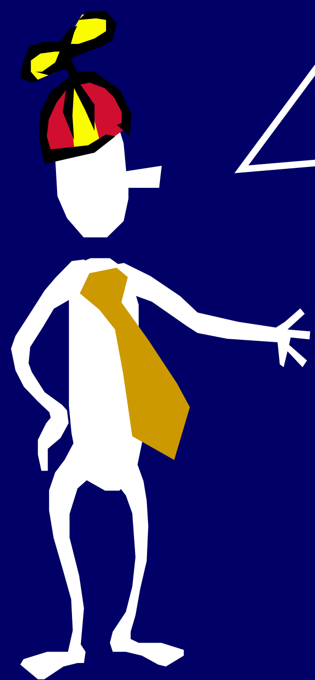
Another Proof of Cayley's Formula

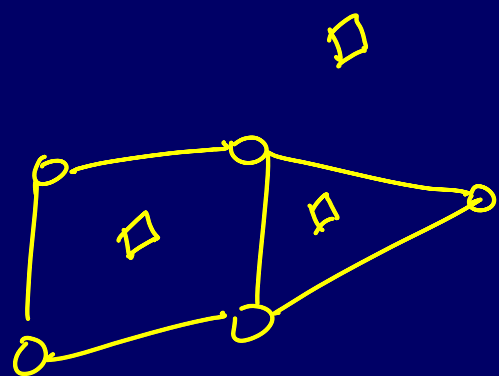
$$T_n = n^{n-2}$$



$$T_{n,k} = k\, n^{n-k-1}$$

A graph is *planar* if it can be drawn in the plane without crossing edges. A *plane graph* is any such drawing, which breaks up the plane into a number *f* of *faces* or regions





$f = 3$

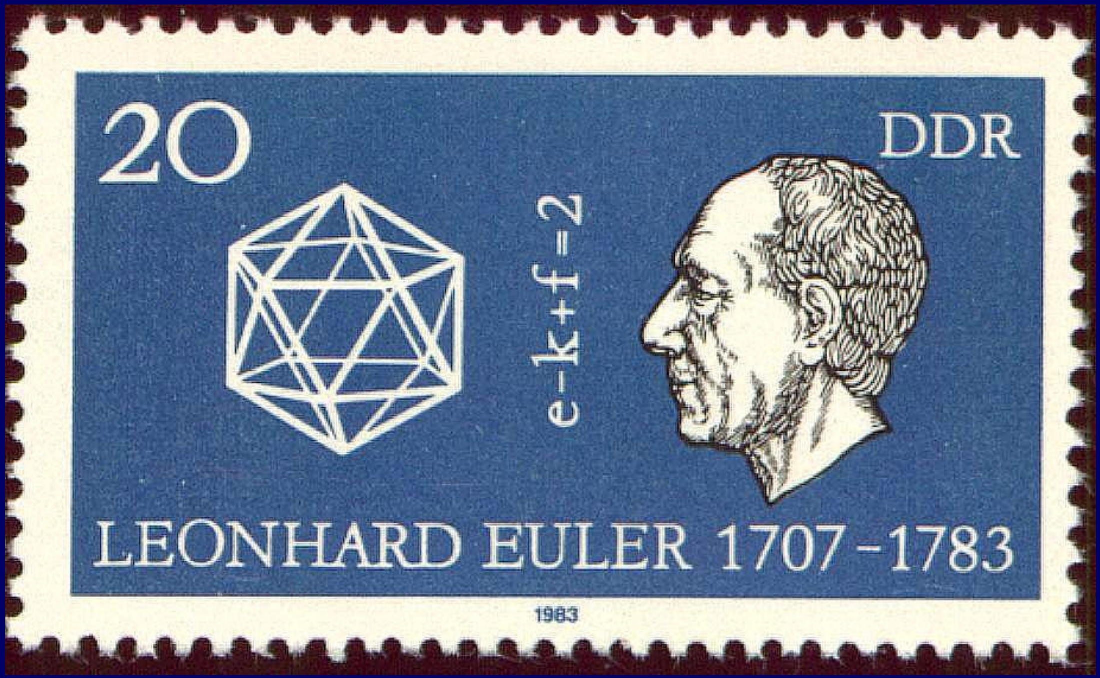
$n = 5$

$e = 6$

$n - e + f = 5 - 6 + 3 = 2$

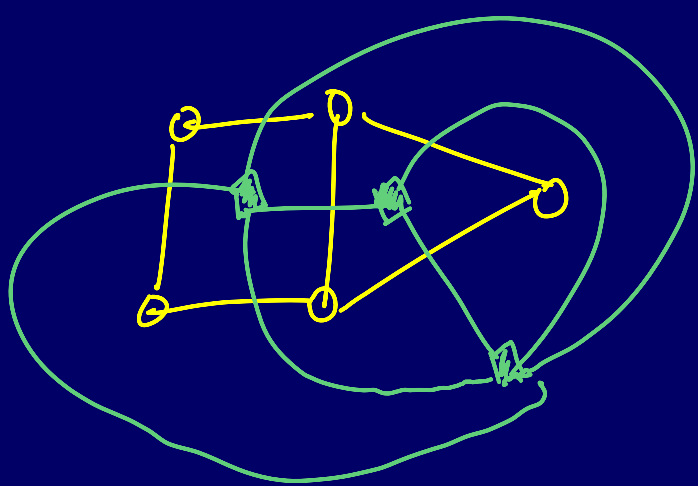
Euler's Formula

If G is a connected plane graph with n vertices, e edges and f faces, then $n - e + f = 2$



Rather than using induction, we'll use the important notion of the *dual graph*

To construct the dual graph, put a vertex into the interior of every face, and connect two such vertices by an edge if there is a common boundary edge between the faces. (Note that the dual graph may have multiple edges between points.)

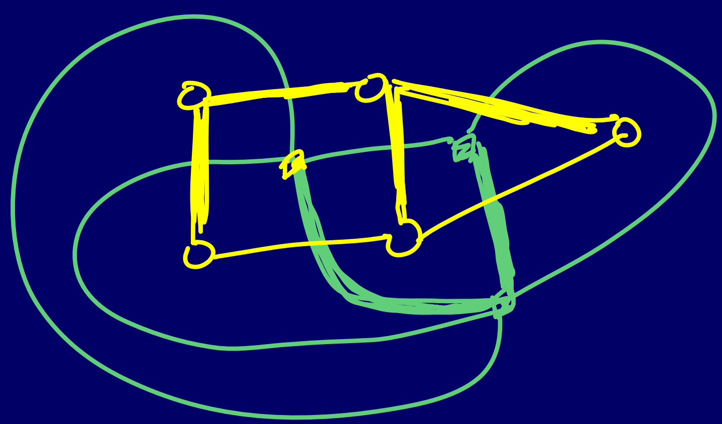


We'll also use the notion of a *spanning tree*

A *spanning tree* of a graph is a subgraph that is also a tree, with the same vertex set as the original graph.

In other words, it's a minimal subgraph that connects all of the vertices.

Euler's Formula: If G is a connected plane graph with n vertices, e edges and f faces, then $n - e + f = 2$



T spanning tree of G

T^* set of edges in G^* corresponding to edges $G \setminus T$

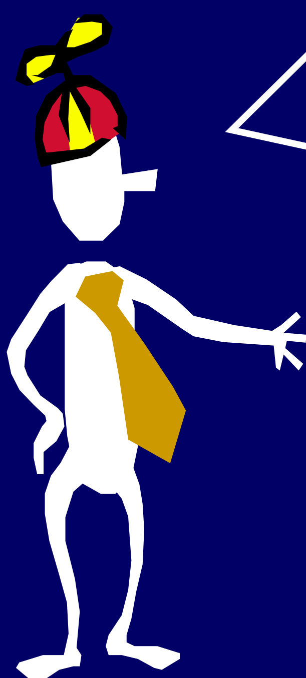
Claim: T^* is a spanning tree of G^*

$n = e_T + 1$ because T is a spanning tree

$f = e_{T^*} + 1$

$$n + f = \underbrace{e_T + e_{T^*}}_{e} + 2$$

Euler's Formula: If G is a connected plane graph with n vertices, e edges and f faces, then $n - e + f = 2$



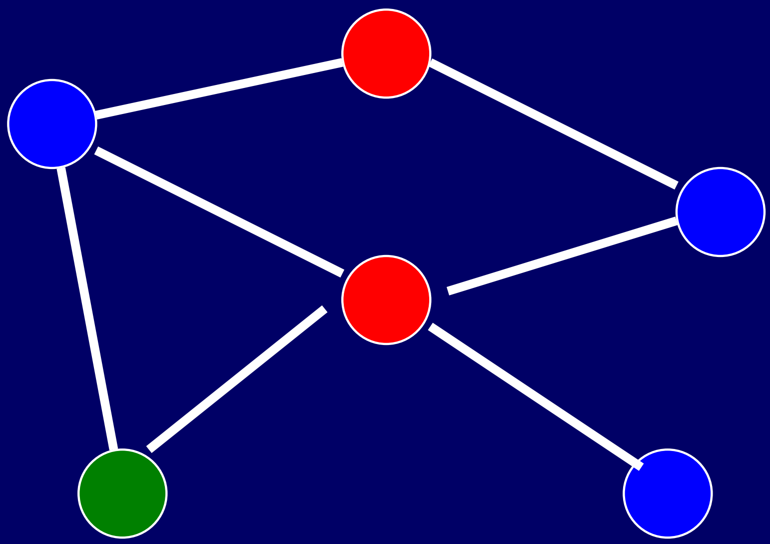
The beauty of Euler's formula is that it yields a *numeric* property from a purely *geometric* or topological property.

Corollary: Let G be a plane graph with $n > 2$ vertices. Then

a) G has a vertex of degree at most 5.
b) G has at most $3n - 6$ edges

Graph Coloring

A coloring of a graph is an assignment of a color to each vertex such that no neighboring vertices have the same color.

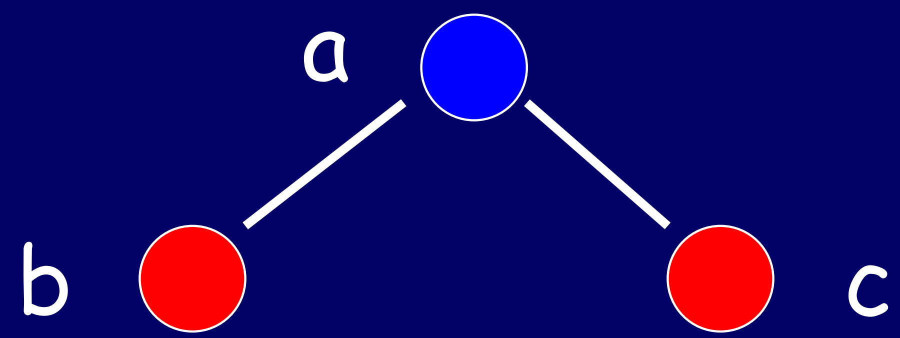


Graph Coloring

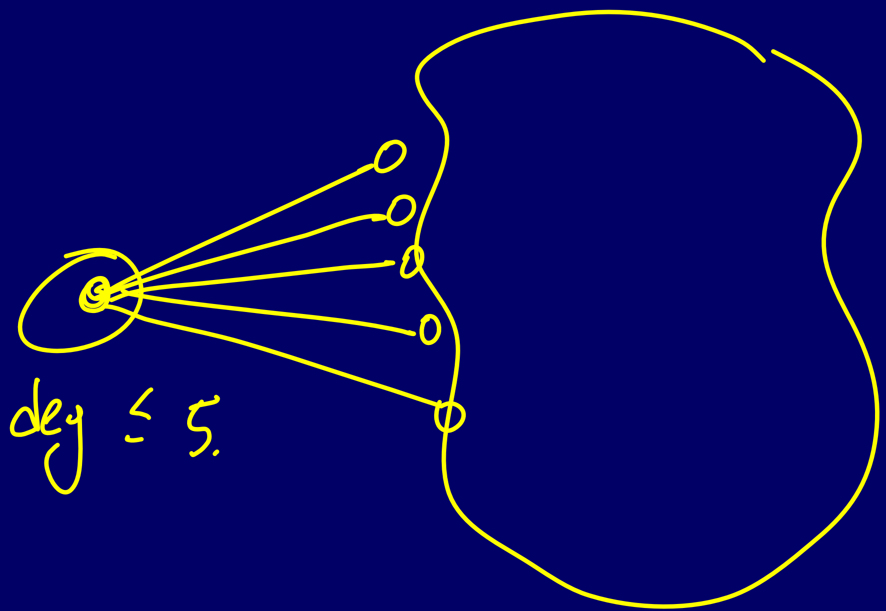
Arises surprisingly often in CS.

Register allocation: assign temporary variables to registers for scheduling instructions. Variables that interfere, or are simultaneously active, cannot be assigned to the same register.

Instructions	Live variables
	a
b = a+2	
	a,b
c = b*b	
	a,c
b = c+1	
	a,b
return a*b	

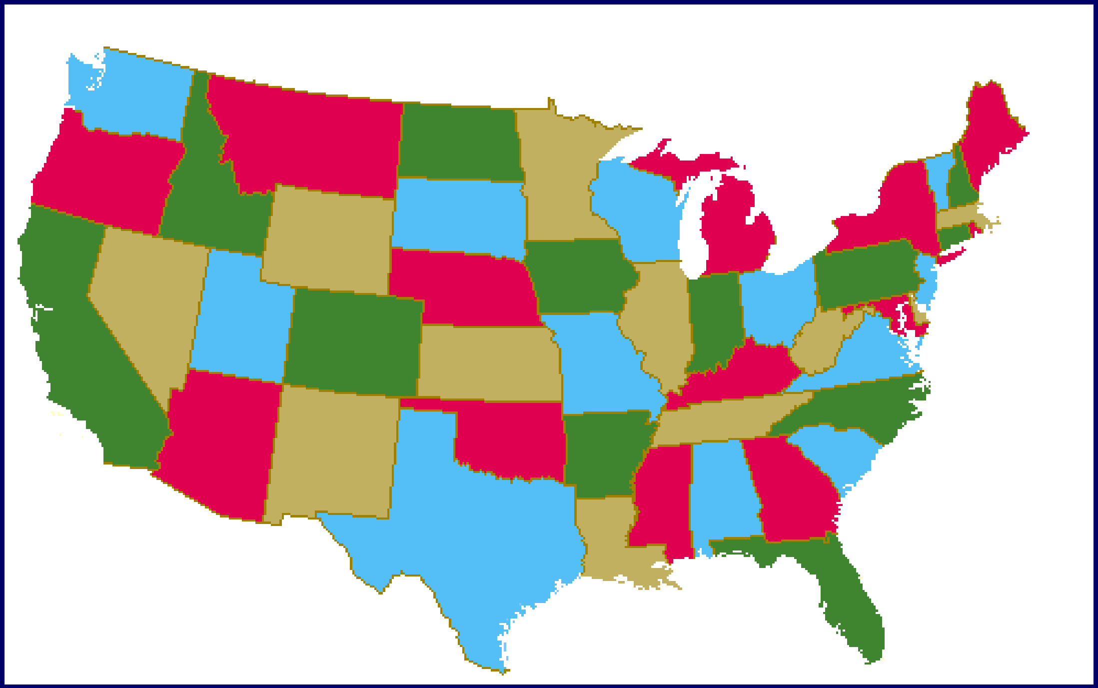


Every plane graph can be 6-colored



Not too difficult to give an inductive proof of 5-colorability, using same fact that some vertex has degree <= 5.

4-color theorem remains challenging



http://www.math.gatech.edu/~thomas/FC/fourcolor.html