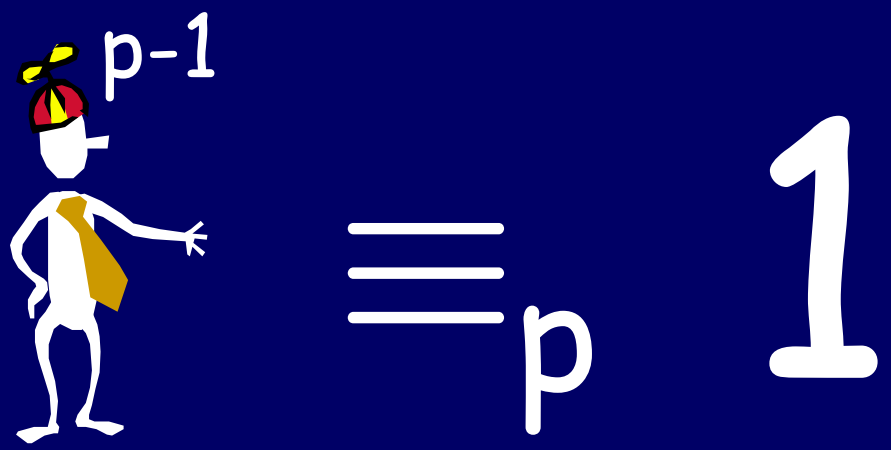
Great Theoretical Ideas In Computer Science

Anupam Gupta CS 15-251 Fall 2006

Lecture 14 Oct 12, 2006 Carnegie Mellon University

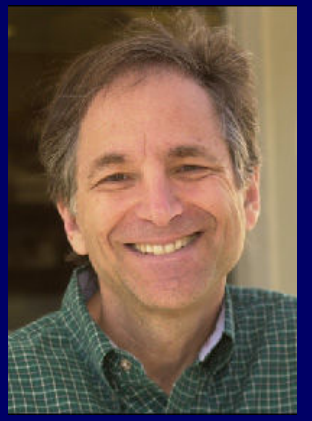
Modular Arithmetic and the RSA Cryptosystem



Starring

Rivest

Shamir



Adleman

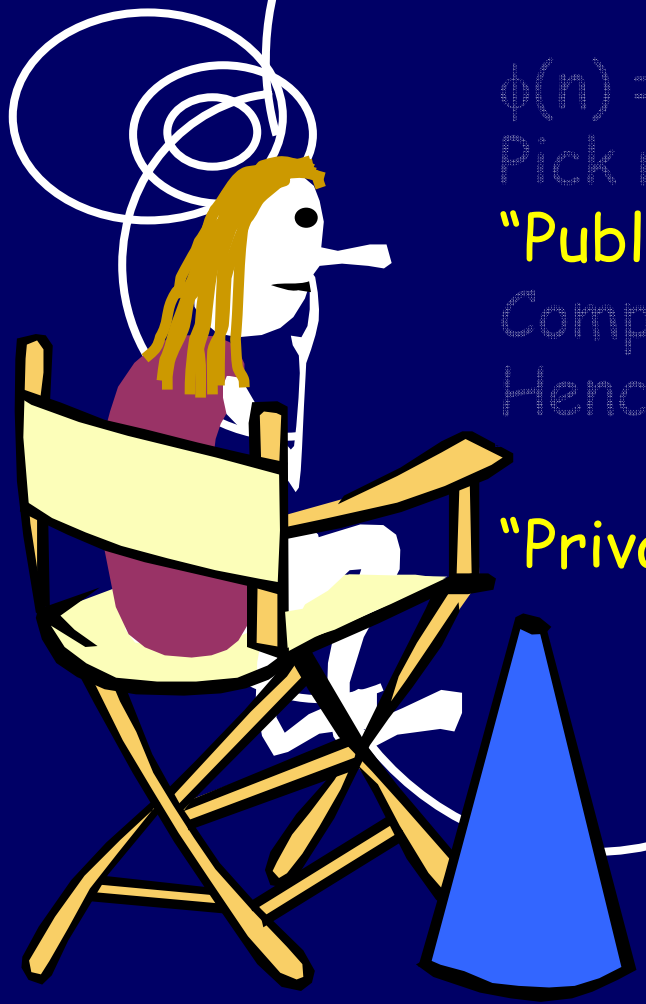
Euler

Fermat

The RSA Cryptosystem

Rivest, Shamir, and Adelman (1978)

RSA is one of the most used cryptographic protocols on the net. Your browser uses it to establish a secure session with a site.



Pick secret, random large primes: p,q

"Publish": n = p*q

$\phi(n) = \phi(p)\,\phi(q) = (p-1)*(q-1)$

Mumbo jumbo…

Pick random $e \in Z^*_{\phi(n)}$

"Publish": e

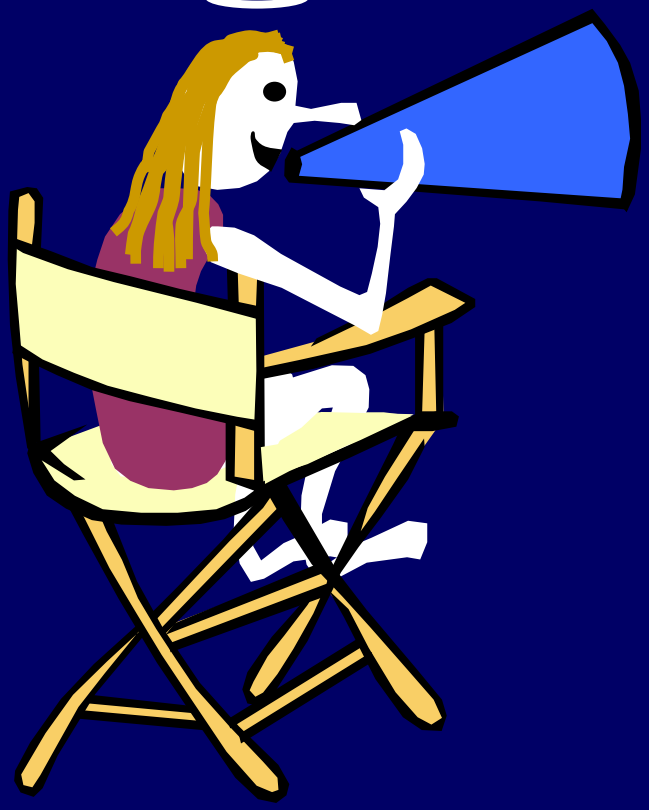
Compute d = inverse of e in $Z^*_{\phi(n)}$

More Mumbo jumbo…

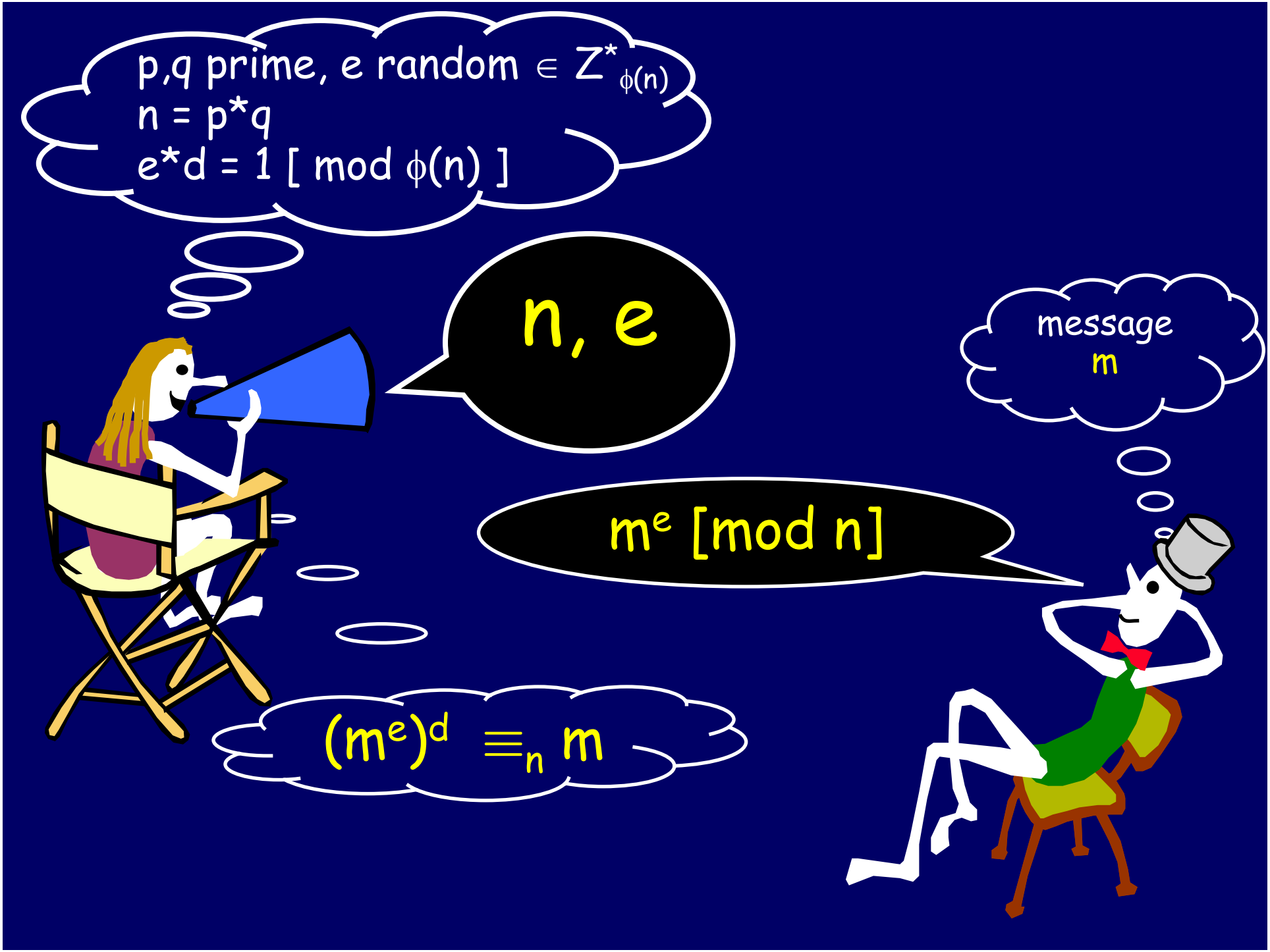
Hence, $e*d = 1 \pmod{\phi(n)}$

"Private Key": d

p,q random primes, e random $\in Z^*_{\phi(n)}$
$n = p*q$
$e*d = 1 \ [\ \text{mod}\ \phi(n)\ ]$



n,e is my public key. Use it to send me a message.


p,q prime, e random ∈ $Z^*_{\phi(n)}$
n = p*q
e*d = 1 [mod ϕ(n)]
n, e
message
m
m^e [mod n]
$(m^e)^d \equiv_n m$

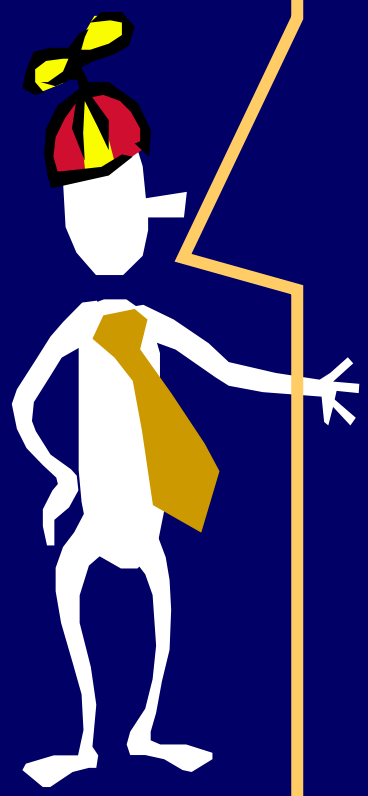
But how does it all work?

What is $\varphi(n)$?
What is $Z_{\varphi(n)}^{*}$?

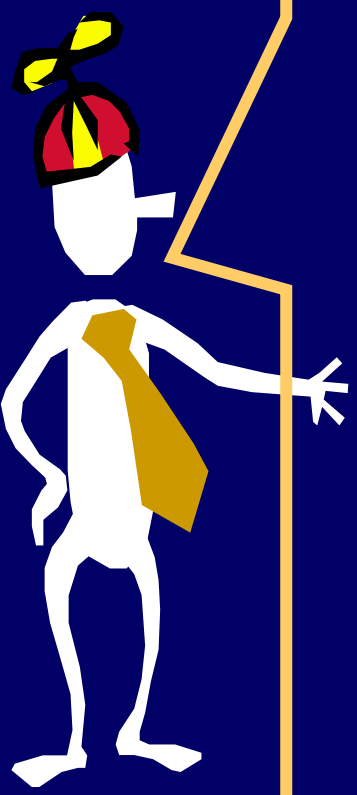
…

Why do all the steps work?

To understand this, we need a little number theory…



$$MAX(a,b) + MIN(a,b) = a+b$$

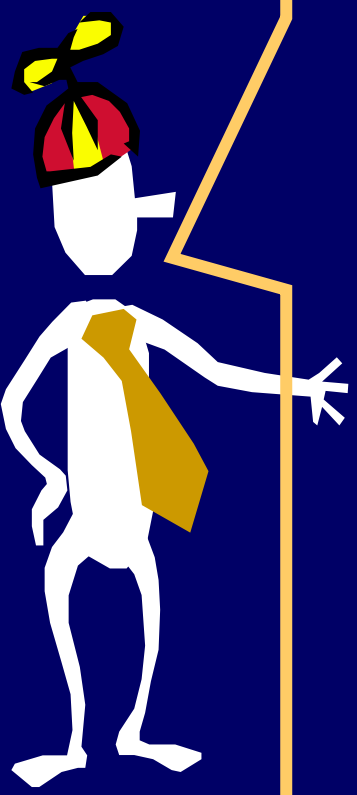


$n|m$ means that m is an integer multiple of n.

We say that "n divides m".

$3|9$ ✓ $2|256$ ✓

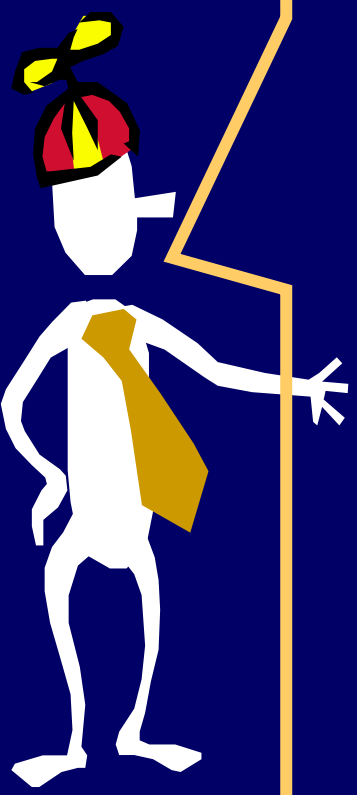
$4|-64$ ✓ $3\nmid 5$



Greatest Common Divisor:

GCD(x,y) =
greatest $k \geq 1$ s.t. $k|x$ and $k|y$.

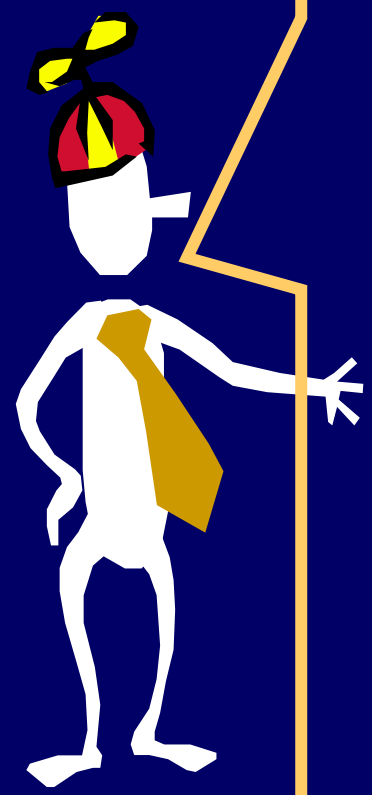
$gcd(12, 18) = 6$



Least Common Multiple:

LCM(x,y) =
smallest $k \geq 1$ s.t. $x|k$ and $y|k$.

$lcm(12, 18) = 36$



Fact:

$$GCD(x,y) \times LCM(x,y) = x \times y$$

$6 \times 36 = 12 \times 18 = 216$

$12 = 2^2 \cdot 3^1$

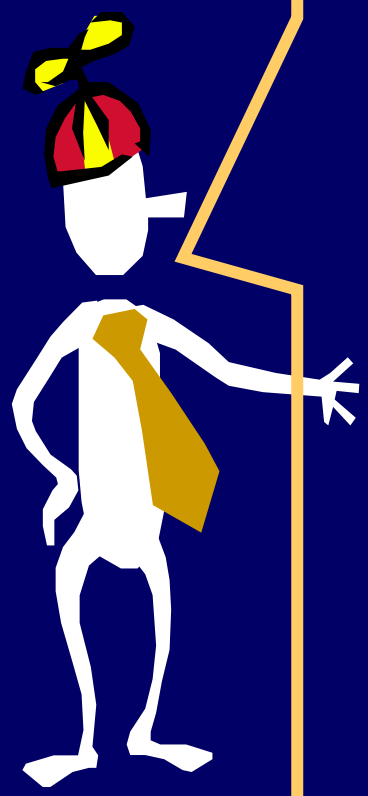
$18 = 2 \cdot 3^2$

$gcd(12,18) = 2^{\min(2,1)} \cdot 3^{\min(1,2)}$

$lcm(12,18) = 2^{\max(2,1)} \cdot 3^{\max(1,2)}$

$2^{1+2} \cdot 3^{1+2}$

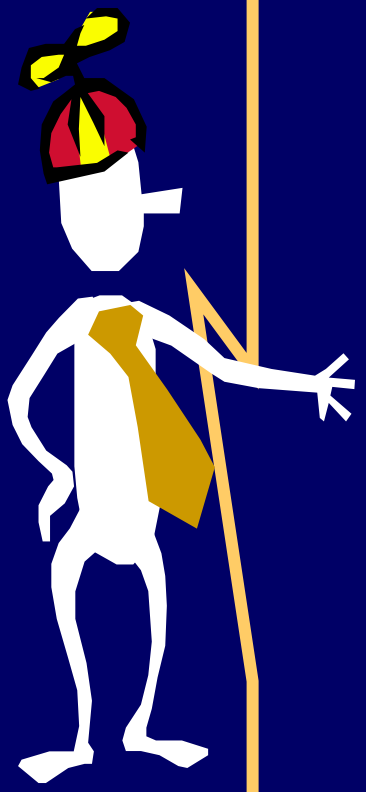
$2^{\min(2,1)+\max(2,1)} \cdot 3^{\min(1,2)+\max(1,2)}$



$$GCD(x,y) \times LCM(x,y) = xy$$

$$MAX(a,b) + MIN(a,b) = a+b$$

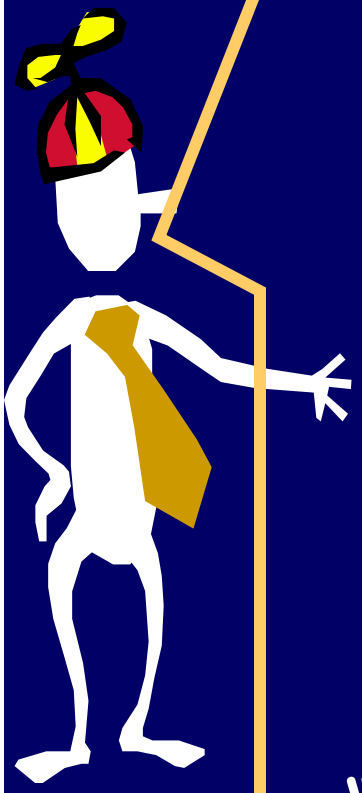
(a mod n) means the remainder when a is divided by n.



If $a = dn + r$ with $0 \leq r < n$

Then $r = (a \bmod n)$

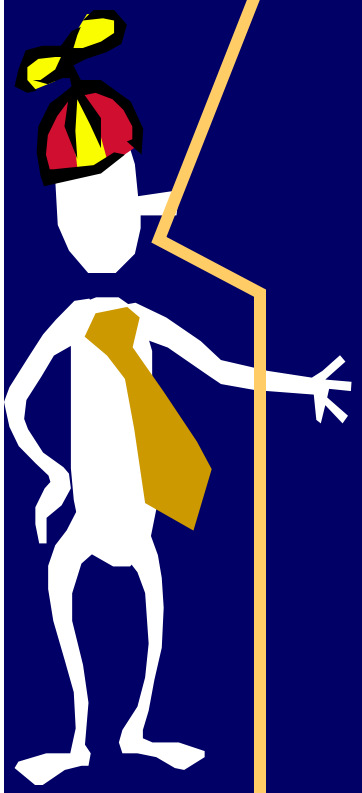
and $d = (a \operatorname{div} n)$



Defn: Modular equivalence of integers a and b

$$a \equiv b \ [\text{mod } n]$$
$$\text{if } (a \bmod n) = (b \bmod n)$$
$$\Leftrightarrow n|(a-b)$$

Written as $a \equiv_n b$, and spoken
"a and b are equivalent modulo n"



$$31 \equiv 81 \ [\mathrm{mod}\ 2]$$

$$31 \equiv_2 81$$

$31 \bmod 2 = 1 = 81 \bmod 2$

$\Leftrightarrow \quad 2 \mid (31-81)$

$\equiv_n$ is an equivalence relation

In other words,

Reflexive:

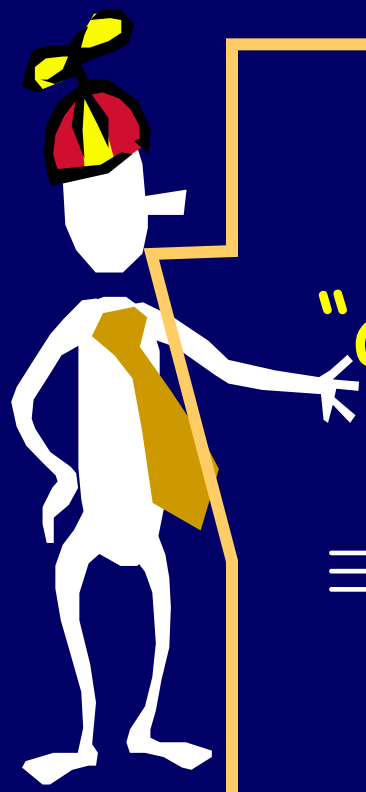
$$a \equiv_n a$$

Symmetric:

$$(a \equiv_n b) \Rightarrow (b \equiv_n a)$$

Transitive:

$$(a \equiv_n b \text{ and } b \equiv_n c) \Rightarrow (a \equiv_n c)$$

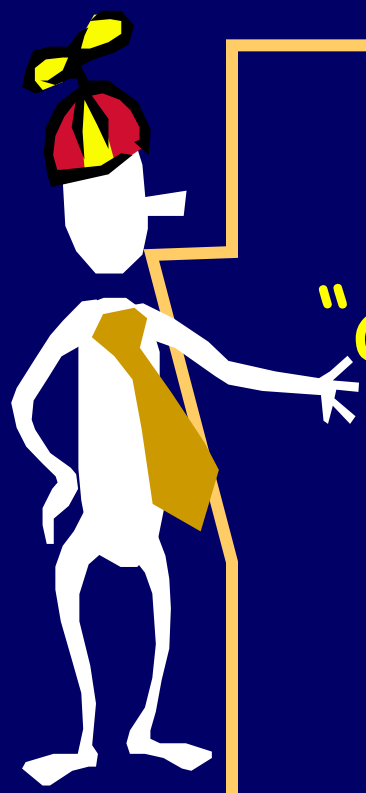


$a \equiv_n b \leftrightarrow n|(a-b)$

"a and b are equivalent modulo n"

$\equiv_n$ induces a natural partition of the integers into n classes.

a and b are said to be in the same "residue class" or "congruence class" exactly when $a \equiv_n b$.



$$a \equiv_n b \leftrightarrow n|(a-b)$$

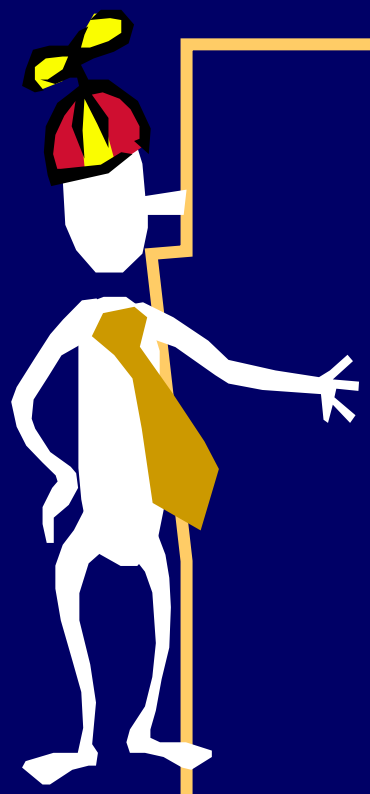
"a and b are equivalent modulo n"

Define

Residue class [i]

=

the set of all integers that are congruent to i modulo n.



Residue Classes Mod 3:

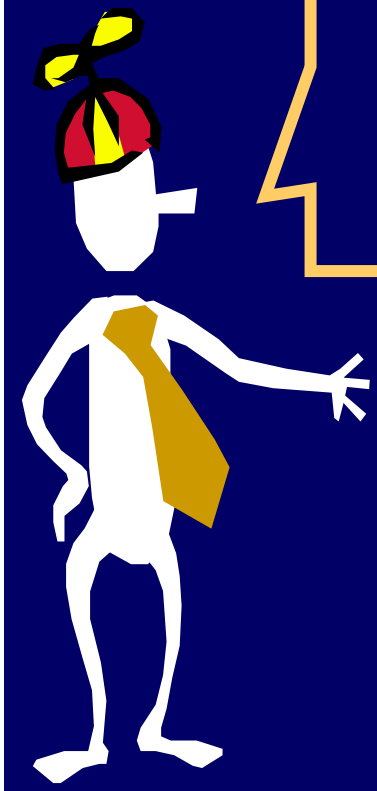
$[0] = \{ \ldots, -6, -3, 0, 3, 6, \ldots \}$
$[1] = \{ \ldots, -5, -2, 1, 4, 7, \ldots \}$
$[2] = \{ \ldots, -4, -1, 2, 5, 8, \ldots \}$
$[-6] = \{ \ldots, -6, -3, 0, 3, 6, \ldots \}$
$[7] = \{ \ldots, -5, -2, 1, 4, 7, \ldots \}$
$[-1] = \{ \ldots, -4, -1, 2, 5, 8, \ldots \}$

Fact: equivalence mod n implies equivalence mod any divisor of n.

If $(x \equiv_n y)$ and $(k|n)$
Then: $x \equiv_k y$

Example: $10 \equiv_6 16 \Rightarrow 10 \equiv_3 16$

If $(x \equiv_n y)$ and $(k|n)$
then $x \equiv_k y$

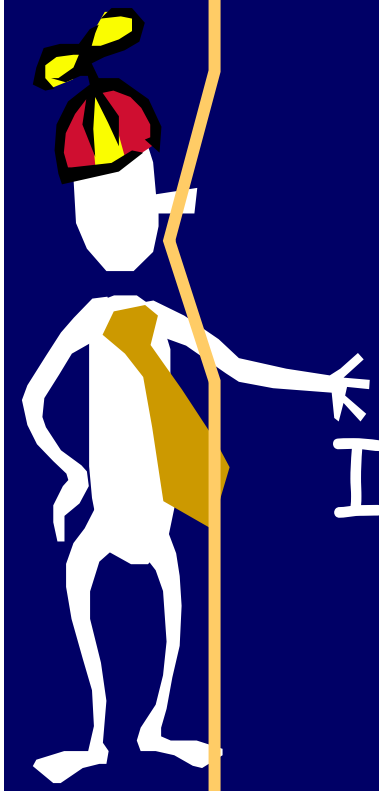


Proof: $x \equiv_n y \Rightarrow n|(x-y)$

but $k|n$

$\Rightarrow k|(x-y)$

$\Rightarrow x \equiv_k y$



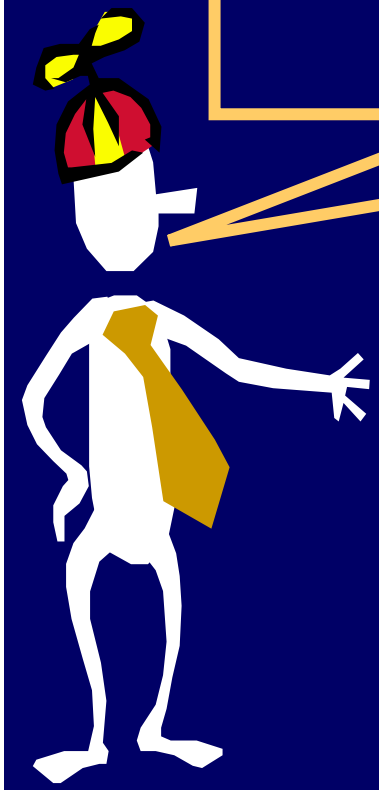
Fundamental lemma of plus, minus, and times modulo n:

If $(x \equiv_n y)$ and $(a \equiv_n b)$. Then

1) $x + a \equiv_n y + b$
2) $x - a \equiv_n y - b$
3) $x * a \equiv_n y * b$

Proof of 3: xa = yb (mod n)

(The other two proofs are similar…)



$x \equiv_n y$

$a \equiv_n b$

$n \mid (x-y) \Rightarrow n \mid a(x-y)$

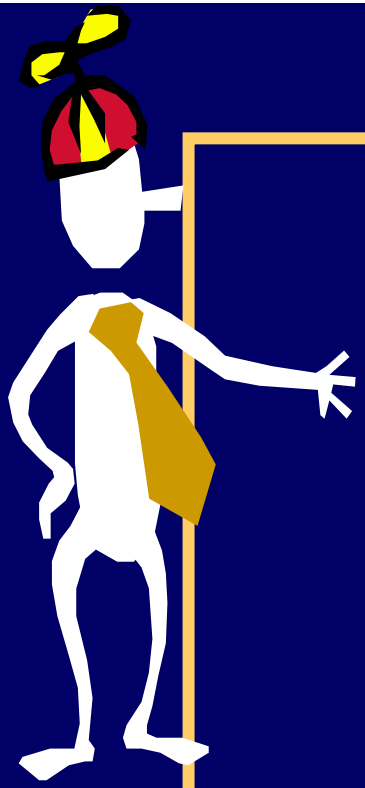
$n \mid (a-b) \Rightarrow n \mid y(a-b)$

$n \mid a(x-y) + y(a-b) \Leftrightarrow n \mid ax - by$

$\Rightarrow ax \equiv_n by$

Fundamental lemma of plus minus, and times modulo n:

When doing plus, minus, and times modulo n, I can at any time in the calculation replace a number with a number in the same residue class modulo n

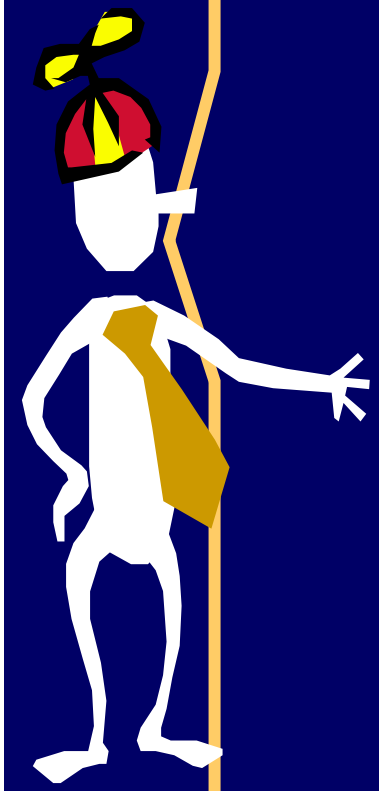


Please calculate:

$249 * 504 \mod 251$

when working mod 251

$-2 * 2 = -4 = 247$

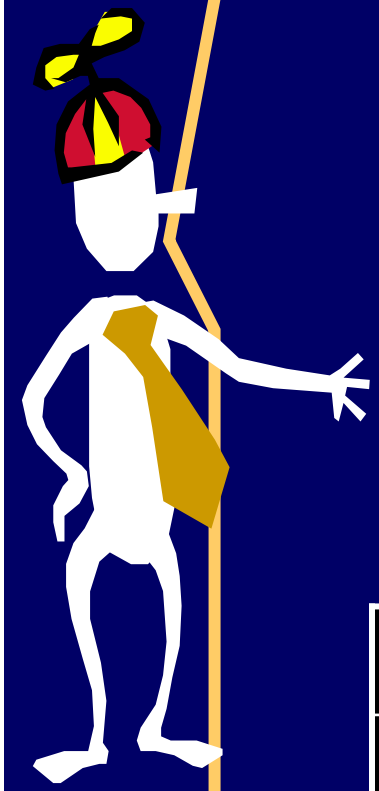


A Unique Representation System Modulo n: [i]

We pick exactly one representative from each residue class.

We do all our calculations using these representatives.

Unique representation system modulo 3



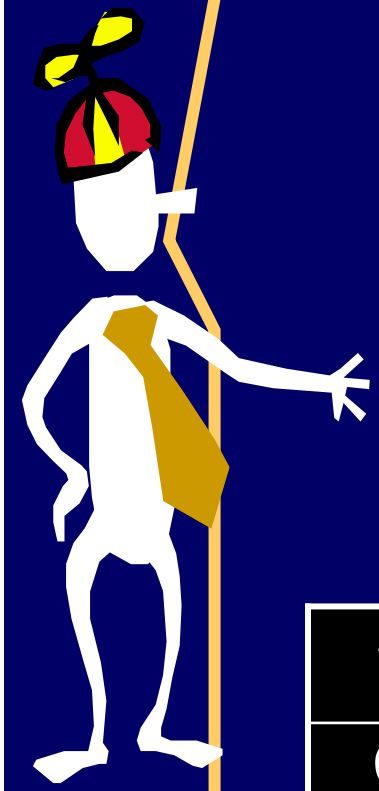
Finite set S = {0, 1, 2}

+ and * defined on S:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

*	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Unique representation system modulo 3

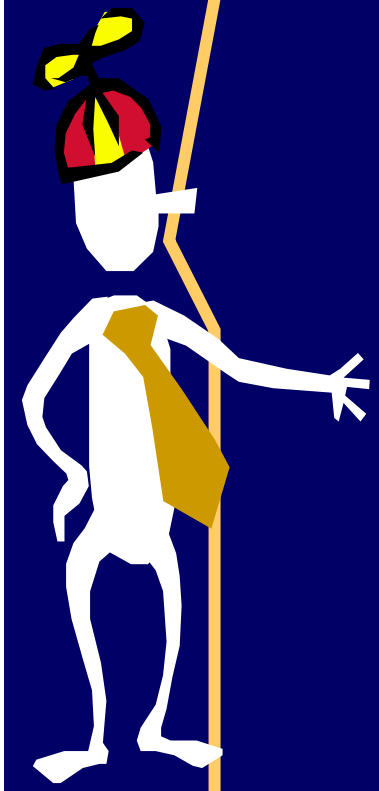


Finite set S = {0, 1, -1}

+ and * defined on S:

+	0	1	-1
0	0	1	-1
1	1	-1	0
-1	-1	0	1

*	0	1	-1
0	0	0	0
1	0	1	-1
-1	0	-1	1



Perhaps the most convenient set of representatives:

The reduced system modulo n:

$$Z_n = \{0, 1, 2, \ldots, n-1\}$$

Define operations $+_n$ and $*_n$:

$$a +_n b = (a+b \bmod n)$$
$$a *_n b = (a*b \bmod n)$$

$$Z_n = \{0, 1, 2, \ldots, n-1\}$$

$a +_n b = (a+b \bmod n)$ $\quad$ $a *_n b = (a*b \bmod n)$

[Closed]

$$x, y \in Z_n \Rightarrow x +_n y \in Z_n$$

[Associative]

$$x, y, z \in Z_n \Rightarrow (x +_n y) +_n z = x +_n (y +_n z)$$

[Commutative]

$$x, y \in Z_n \Rightarrow x +_n y = y +_n x$$

$$Z_n = \{0, 1, 2, \ldots, n-1\}$$

$a +_n b = (a+b \bmod n)$ $\quad$ $a *_n b = (a*b \bmod n)$

[Closed]

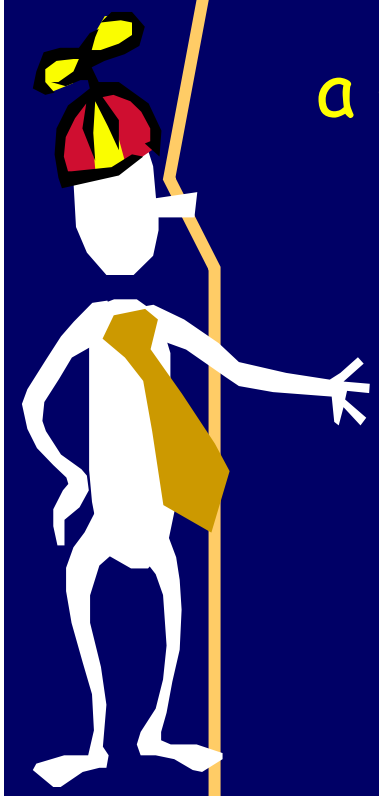
$$x, y \in Z_n \Rightarrow x *_n y \in Z_n$$

[Associative]

$$x, y, z \in Z_n \Rightarrow (x *_n y) *_n z = x *_n (y *_n z)$$

[Commutative]

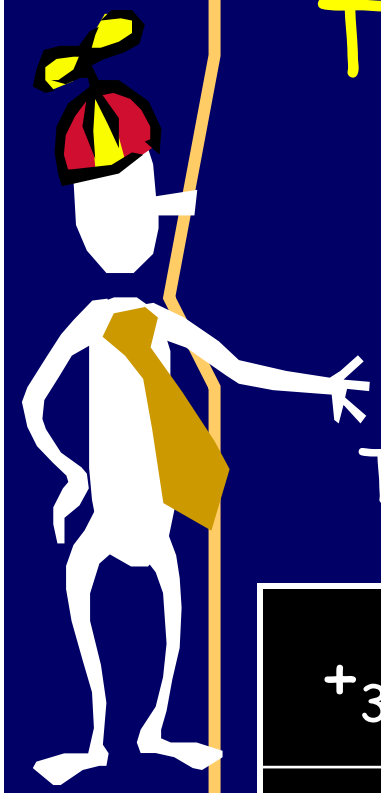
$$x, y \in Z_n \Rightarrow x *_n y = y *_n x$$



$Z_n = \{0, 1, 2, \ldots, n-1\}$

$a +_n b = (a+b \bmod n)$ $\quad$ $a *_n b = (a*b \bmod n)$

$+_n$ and $*_n$ are
commutative,
associative
binary operators
from $Z_n \times Z_n \rightarrow Z_n$:



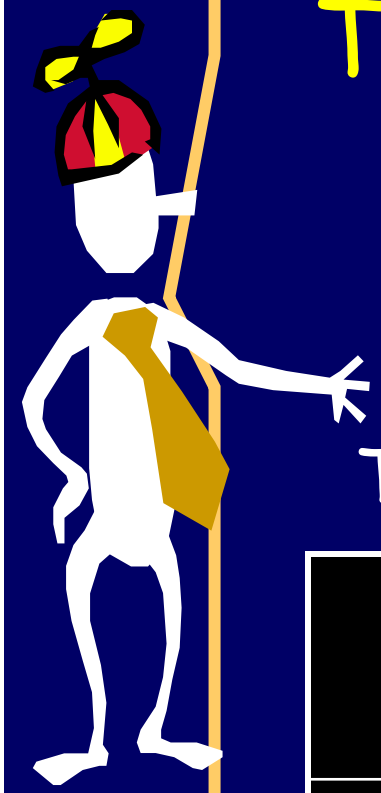
The reduced system modulo 3

$$Z_3 = \{0, 1, 2\}$$

Two binary, associative operators on Z_3:

$+_3$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$*_3$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1



The reduced system modulo 2

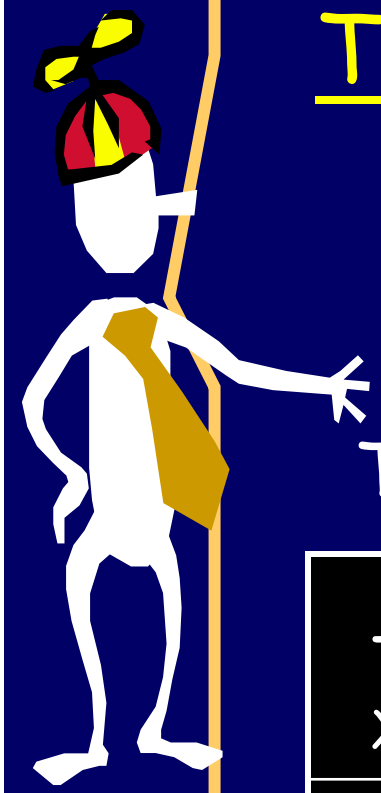
$$Z_2 = \{0, 1\}$$

Two binary, associative operators on Z_2:

$+_2$	0	1
0	0	1
1	1	0

、

$*_2$	0	1
0	0	0
1	0	1



The Boolean interpretation of Z_2

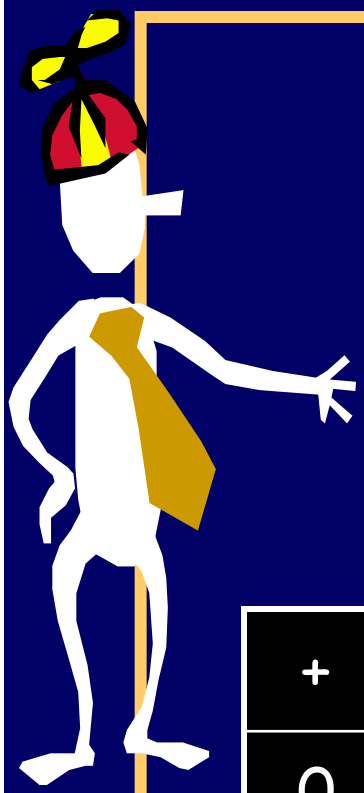
$$Z_2 = \{0, 1\}$$

Two binary, associative operators on Z_2:

$+_2$ XOR	0	1
0	0	1
1	1	0

、

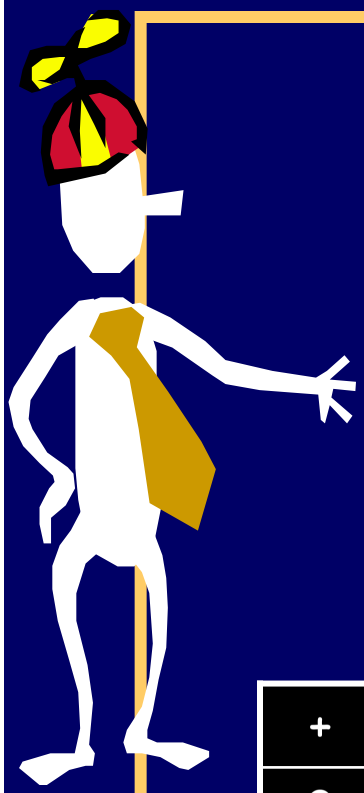
$*_2$ AND	0	1
0	0	0
1	0	1



The reduced system
$Z_4 = \{0, 1, 2, 3\}$

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

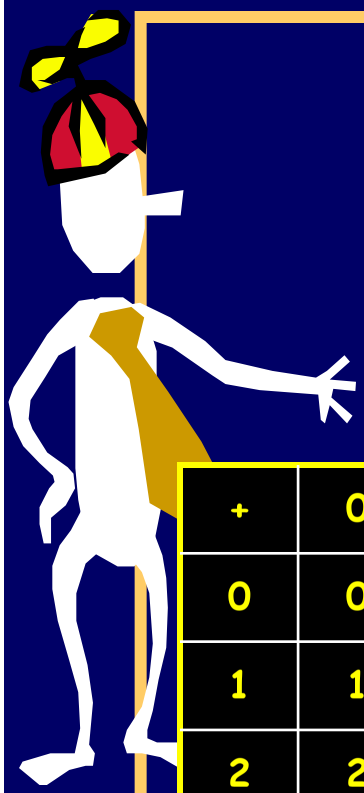
*	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1



The reduced system $Z_5 = \{0,1,2,3,4\}$

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

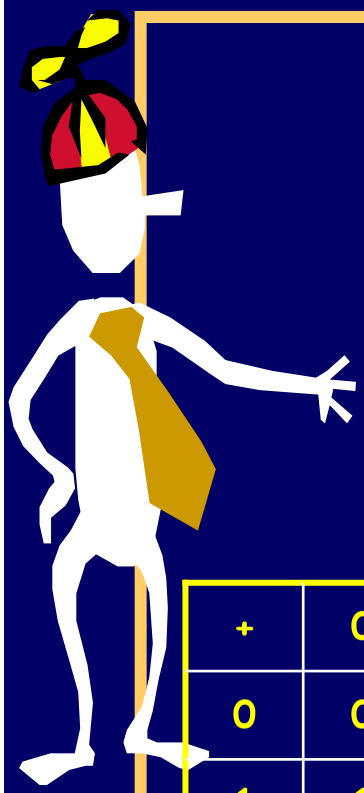
*	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1



The reduced system $Z_6 = \{0,1,2,3,4,5\}$

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

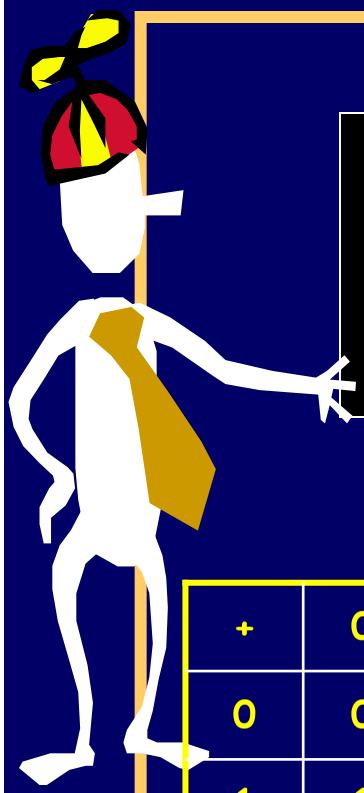
*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1



The reduced system $Z_6 = \{0,1,2,3,4,5\}$

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

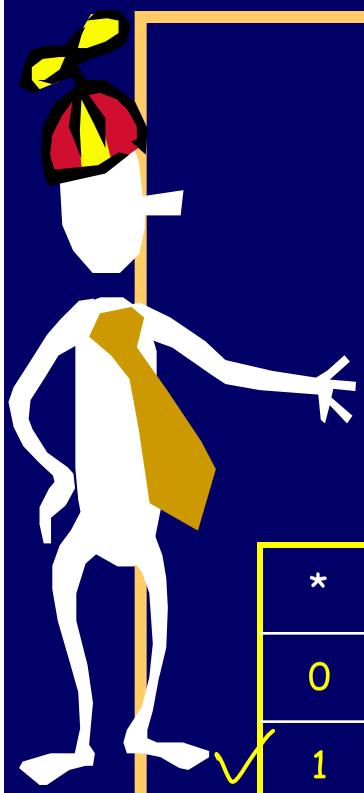
An operator has the permutation property if each row and each column has a permutation of the elements.



For every n, $+_n$ on Z_n has the permutation property

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

An operator has the permutation property if each row and each column has a permutation of the elements.



What about multiplication? Does $*_6$ on Z_6 have the permutation property?

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

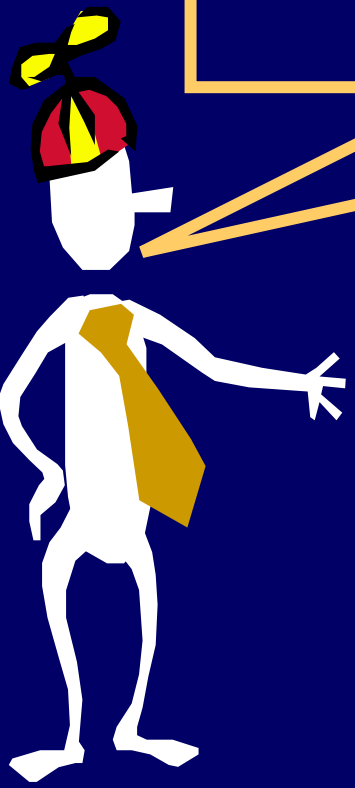
An operator has the permutation property if each row and each column has a permutation of the elements.

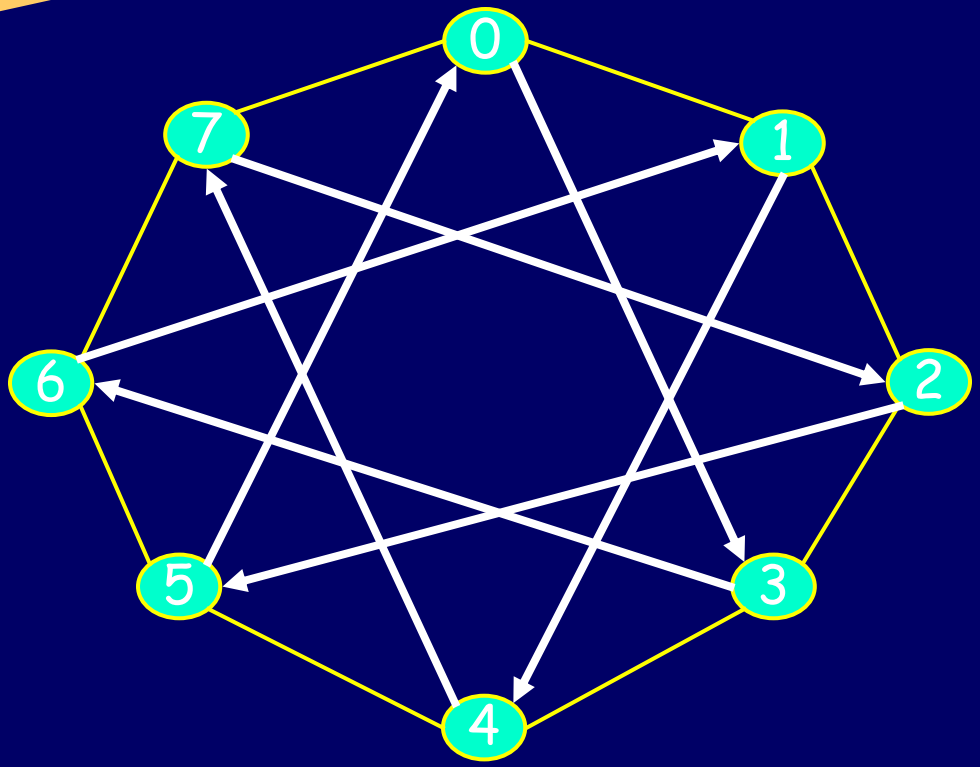
What about $*_8$ on Z_8?

*	0	1	2	3	4	5	6	7	
0	0	0	0	0	0	0	0	0	x
1	0	1	2	3	4	5	6	7	✓
2	0	2	4	6	0	2	4	6	x
3	0	3	6	1	4	7	2	5	✓
4	0	4	0	4	0	4	0	4	x
5	0	5	2	7	4	1	6	3	✓
6	0	6	4	2	0	6	4	2	x
7	0	7	6	5	4	3	2	1	✓

Which rows have the permutation property?

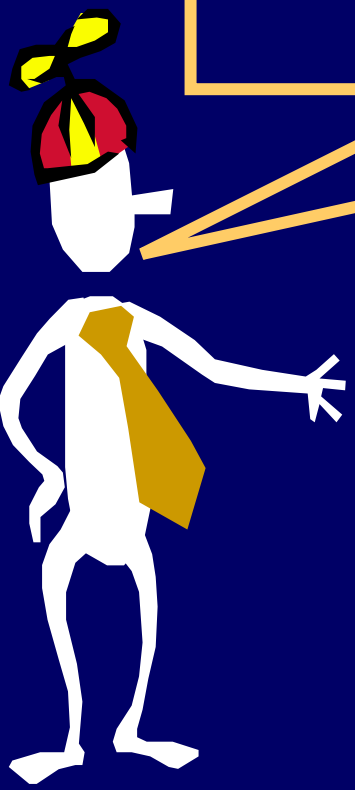
There are exactly 8 distinct multiples of 3 modulo 8.

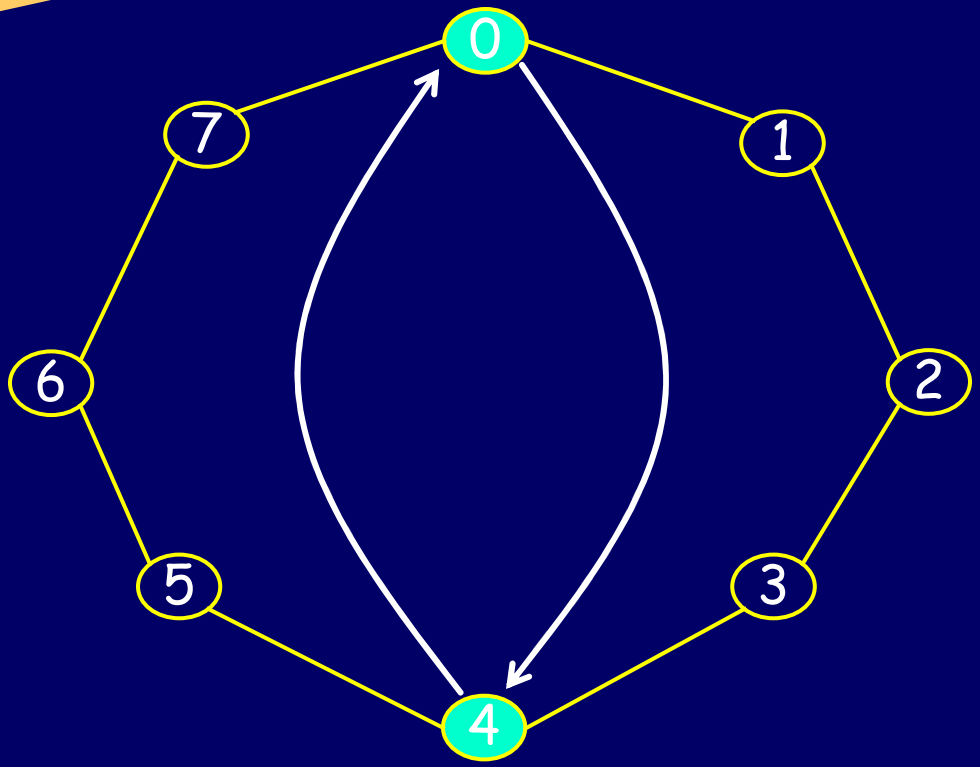




hit all numbers $\Leftrightarrow$ row 3 has the "permutation property"

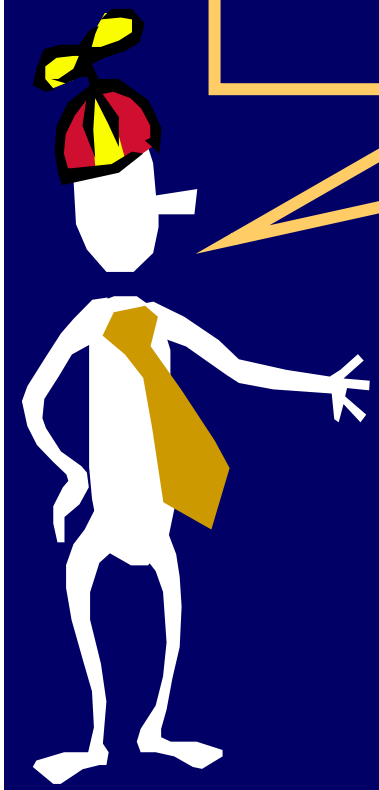
There are exactly 2 distinct multiples of 4 modulo 8

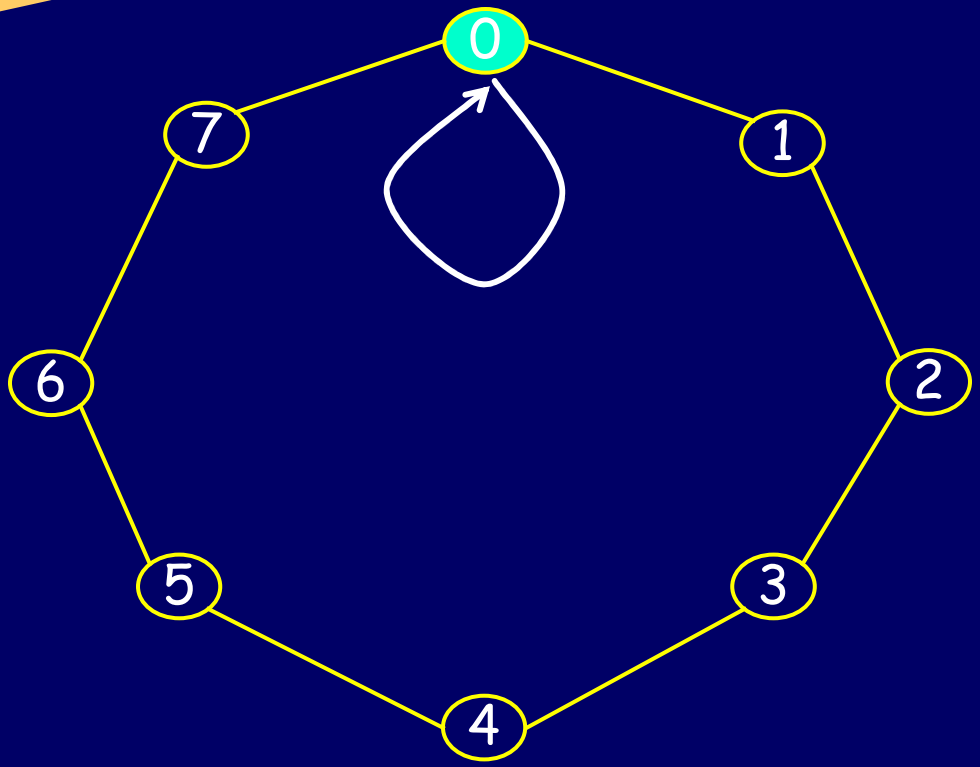




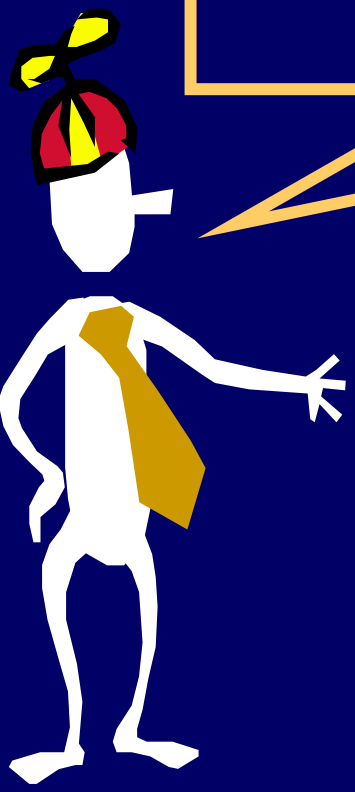
row 4 does not have "permutation property" for $*_8$ on Z_8

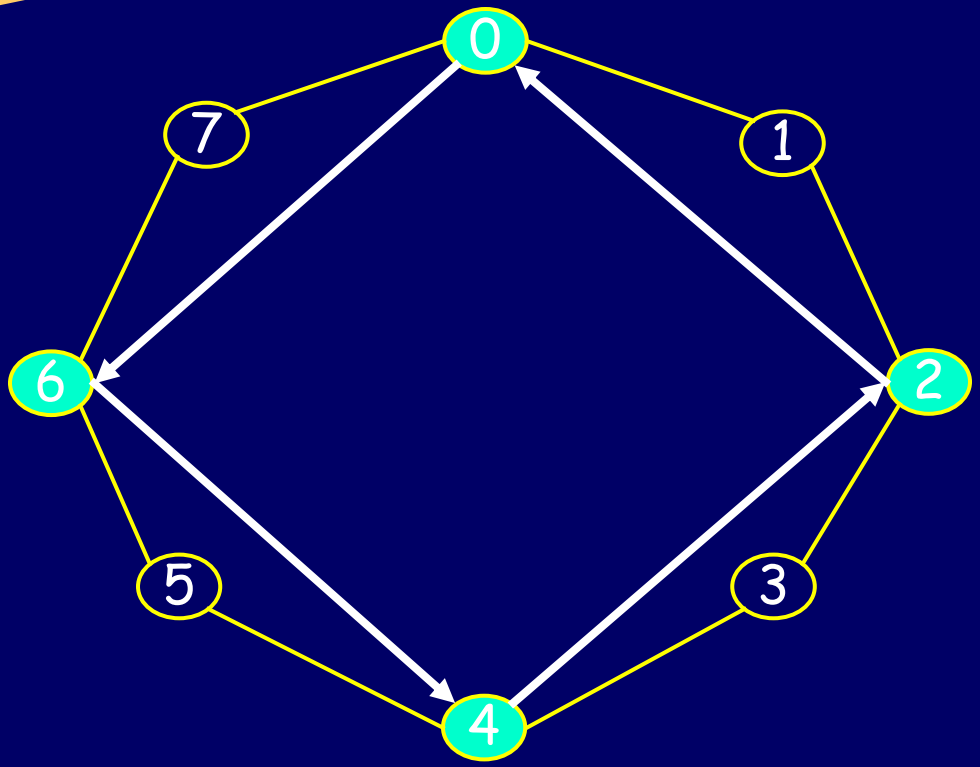
There is exactly 1 distinct multiple of 8 modulo 8

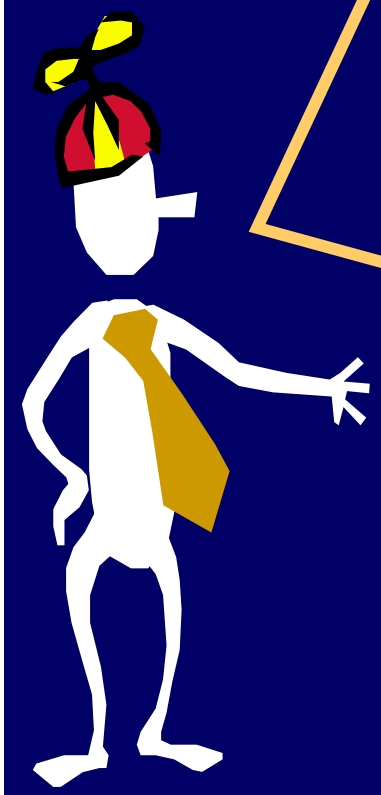




There are exactly 4 distinct multiples of 6 modulo 8







There are exactly
LCM(n,c)/c = n/GCD(c,n)
distinct multiples of c modulo n

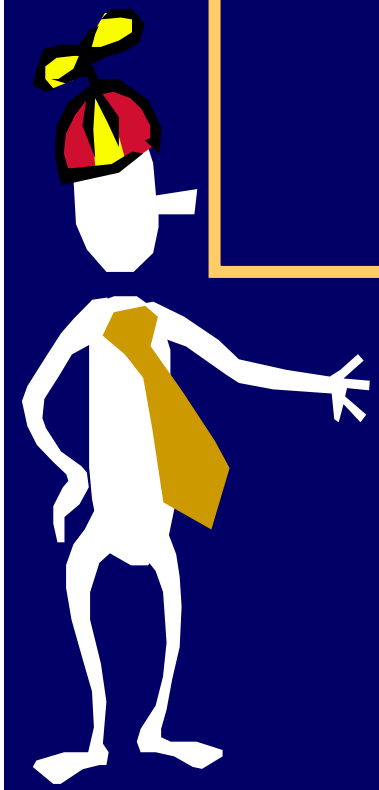
and hence

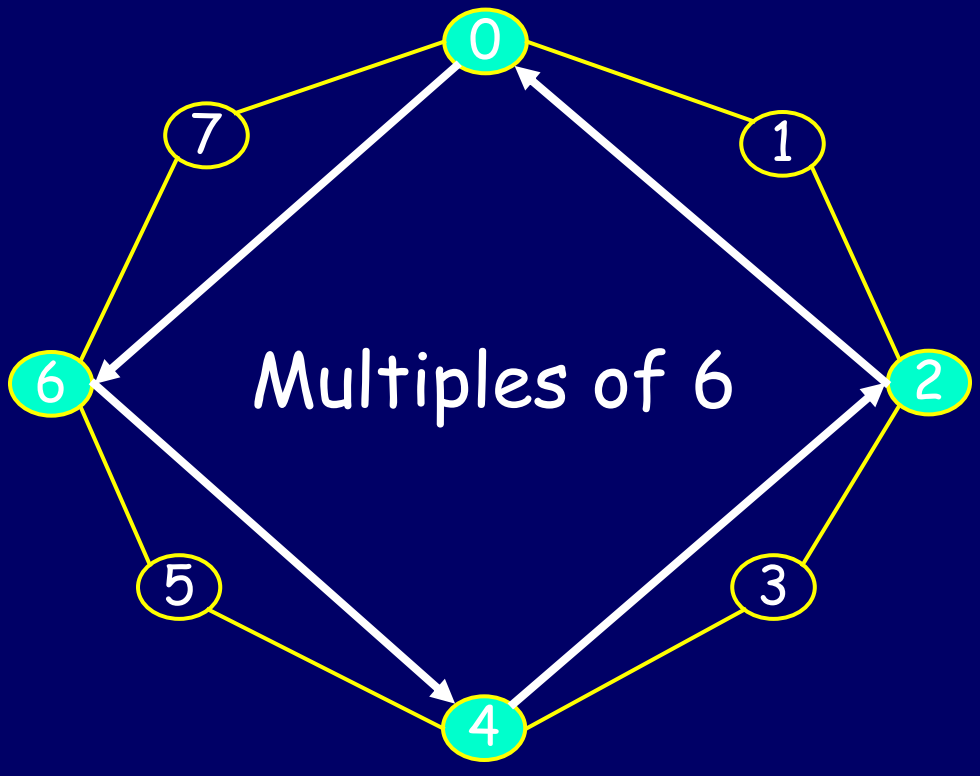
values of c with GCD(c,n) = 1
have the permutation property
for $*_n$ on Z_n

The multiples of c modulo n is the set:

$\{0, c, c +_n c, c +_n c +_n c, \ldots.\}$

$= \{kc \bmod n \mid 0 \le k \le n-1\}$





Theorem: There are exactly
$k = n/GCD(c,n) = LCM(c,n)/c$
distinct multiples of c modulo n: $\{ c*i \bmod n \mid 0 \le i < k \}$

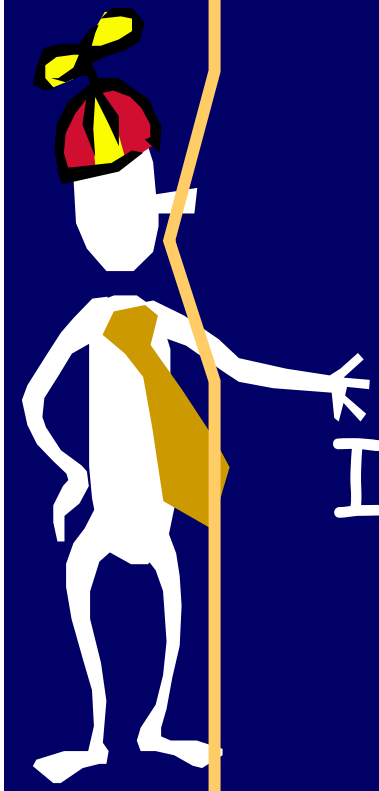
Clearly, $c/GCD(c,n) \geq 1$ is a whole number

$ck = n\,[c/GCD(c,n)] \equiv_n 0$

$\Rightarrow$ There are $\le k$ distinct multiples of c mod n:
$c*0, c*1, c*2, \ldots, c*(k-1)$

Also, k = all the factors of n missing from c
$\Rightarrow cx \equiv_n cy \leftrightarrow n|c(x-y) \Rightarrow k|(x-y) \Rightarrow x-y \geq k$

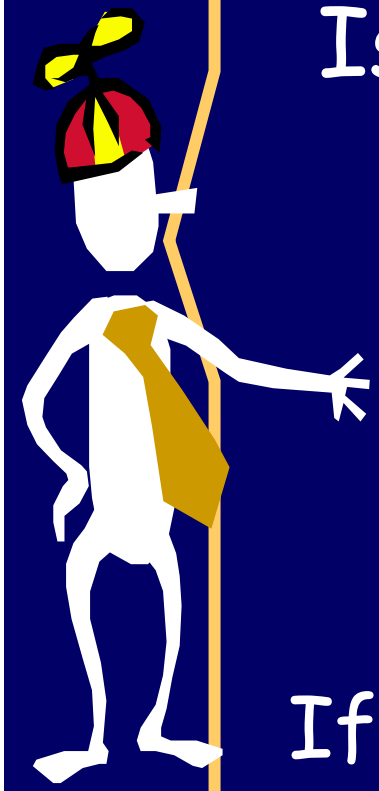
There are $\geq k$ multiples of c. Hence exactly k.



Fundamental lemma of plus, minus, and times modulo n:

If $(x \equiv_n y)$ and $(a \equiv_n b)$. Then

1) $x + a \equiv_n y + b$
2) $x - a \equiv_n y - b$
3) $x * a \equiv_n y * b$



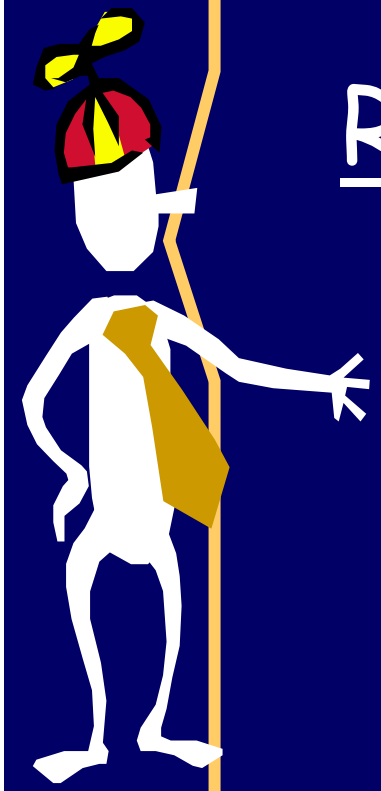
Is there a fundamental lemma of division modulo n?

$cx \equiv_n cy \Rightarrow x \equiv_n y$?

Of course not!

If $c=0 \pmod n$, $cx \equiv_n cy$ for all x and y.

Canceling the c is like dividing by zero.



Let's fix that!

Repaired fundamental lemma of division modulo n?

if $c \neq 0$ [mod n], then

$cx \equiv_n cy \Rightarrow x \equiv_n y$?

$6*3 \equiv_{10} 6*8$, but not $3 \equiv_{10} 8$.

$2*2 \equiv_6 2*5$, but not $2 \equiv_6 5$.

Bummer!

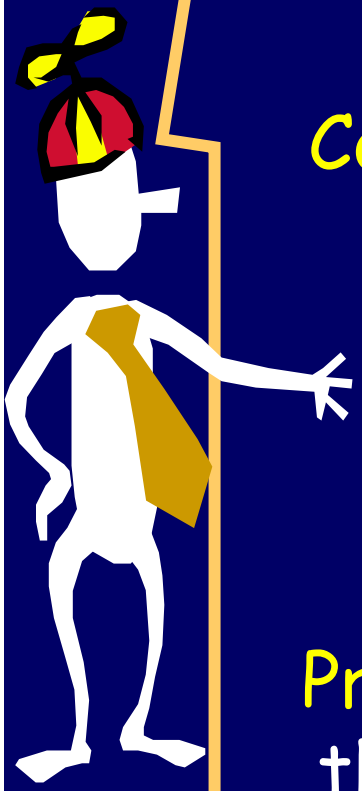
When can I divide by c?

Theorem: There are exactly n/GCD(c.n) distinct multiples of c modulo n.

Corollary: If GCD(c,n) > 1, then the number of multiples of c is less than n.

Corollary: If GCD(c,n) > 1 then you can't always divide by c.

Proof: There must exist distinct x,y<n such that c*x=c*y (but x≠y). Hence cant divide.



Fundamental lemma of division modulo n:

if GCD(c,n)=1, then $ca \equiv_n cb \Rightarrow a \equiv_n b$

Proof:

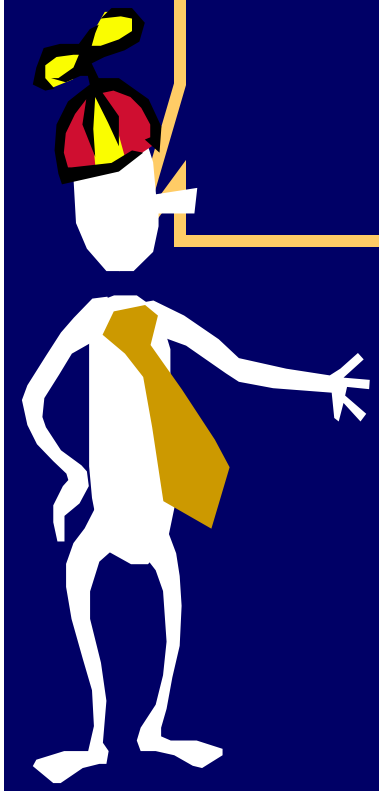
$n \mid ca - cb \Rightarrow n \mid c(a-b)$

but $gcd(c, n) = 1$

$\Rightarrow n \mid (a-b) \Rightarrow a \equiv_n b.$

Corollary for general c:

$cx \equiv_n cy \Rightarrow x \equiv_{n/GCD(c,n)} y$



$n \mid cx - cy = c(x-y)$

$$\frac{n}{\gcd(c,n)} \mid \frac{c}{\gcd(c,n)} (x-y)$$

$$\Rightarrow \quad x = y \pmod{\frac{n}{\gcd(c,n)}}$$

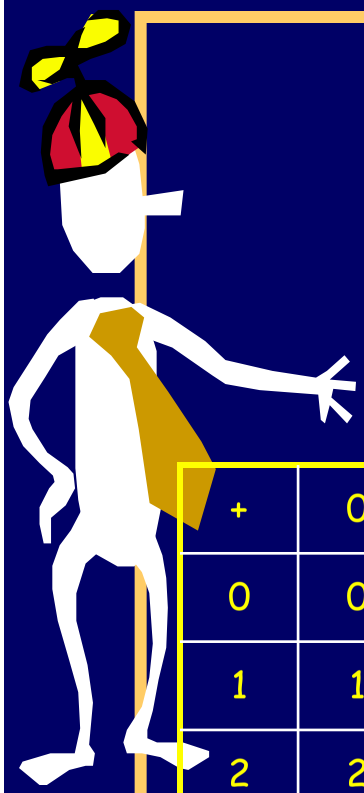
Fundamental lemma of division modulo n.

If GCD(c,n)=1, then $ca \equiv_n cb \Rightarrow a \equiv_n b$

Consider the set

$$Z_n^* = \{x \in Z_n \mid GCD(x,n) = 1\}$$

Multiplication over this set Z_n^* will have the cancellation property.



$Z_6 = \{0, 1, 2, 3, 4, 5\}$

$Z_6^* = \{1, 5\}$

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

What are the properties of Z_n^*

For $*_n$ on Z_n we showed the following properties:

[Closure]

$$x, y \in Z_n \Rightarrow x *_n y \in Z_n$$

[Associativity]

$$x, y, z \in Z_n \Rightarrow (x *_n y) *_n z = x *_n (y *_n z)$$

[Commutativity]

$$x, y \in Z_n \Rightarrow x *_n y = y *_n x$$

What about $*_n$ on Z_n^* ?

All these 3 properties hold for $*_n$ on Z_n^*.

Let's show "closure": $x,y \in Z_n^* \Rightarrow x *_n y \in Z_n^*$

First, a simple fact:

Suppose GCD(x,n) = 1 and GCD(y,n) = 1

Let z = xy. Clearly, GCD(z, n) = 1.

Also, define z′ = (xy mod n). Then GCD(z′,n)=1

All these 3 properties hold for $*_n$ on Z_n^*.

Let's show "closure": $x,y \in Z_n^* \Rightarrow x *_n y \in Z_n^*$

Proof: Let $z = xy$. Let $z' = z \bmod n$. Then $z = z' + kn$.

Suppose z' not in Z_n^*. Then $GCD(z', n) > 1$.
and hence $GCD(z, n) > 1$.

Hence there exists a prime $p>1$ s.t. $p|z'$ and $p|n$.

$p|z \Rightarrow p|x$ or $p|y$. (say $p|x$)

Hence $p|n$, $p|x$, so $GCD(x,n) > 1$.

Contradiction of $x \in Z_n^*$

The properties of Z_n^*

For $*_n$ on Z_n^* the following properties hold:

[Closure]

$$x, y \in Z_n^* \Rightarrow x *_n y \in Z_n^*$$

[Associativity]

$$x, y, z \in Z_n^* \Rightarrow (x *_n y) *_n z = x *_n (y *_n z)$$

[Commutativity]

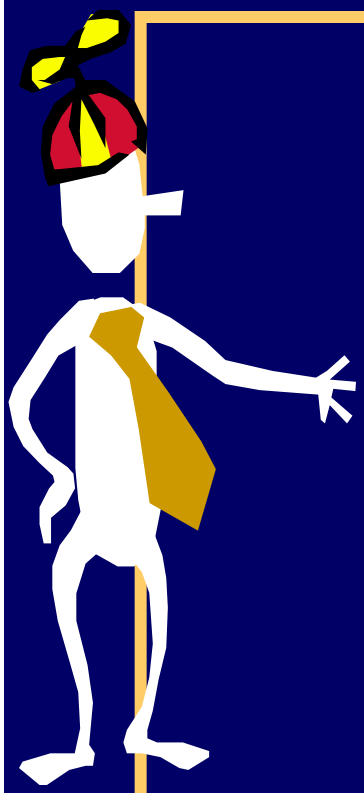
$$x, y \in Z_n^* \Rightarrow x *_n y = y *_n x$$

$$Z_{12}^{*} = \{0 \le x < 12 \mid \gcd(x,12) = 1\}$$
$$= \{1,5,7,11\}$$

$*_{12}$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

Z_{15}^*

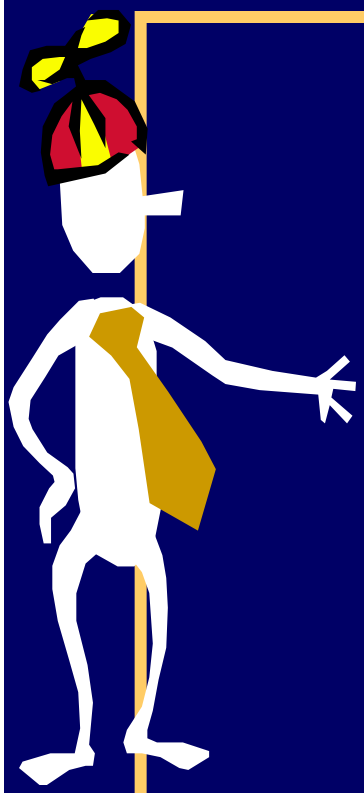
*	1	2	4	7	8	11	13	14
1	1	2	4	7	8	11	13	14
2	2	4	8	14	1	7	11	13
4	4	8	1	13	2	14	7	11
7	7	14	13	4	11	2	1	8
8	8	1	2	11	4	13	14	7
11	11	7	14	2	13	1	8	4
13	13	11	7	1	14	8	4	2
14	14	13	11	8	7	4	2	1



$Z_5^* = \{1,2,3,4\}$ $= Z_5 \setminus \{0\}$

$*_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

For all primes p, $Z_p^* = Z_p \setminus \{0\}$,
since all $0 < x < p$ satisfy $\gcd(x,p) = 1$



(Totient)

Euler Phi Function $\phi(n)$

Define $\phi(n)$ = size of Z_n^*
= number of $1 \le k < n$ that
are relatively prime to n.

$\phi(6) = 2$

p prime $\Rightarrow Z_p^* = \{1,2,3,\ldots,p-1\}$
$\Rightarrow \Phi(p) = p-1$

$\phi(35) = ?$

$$Z_{12}^{*} = \{0 \le x < 12 \mid \gcd(x,12) = 1\}$$
$$= \{1,5,7,11\}$$

$$\phi(12) = 4$$

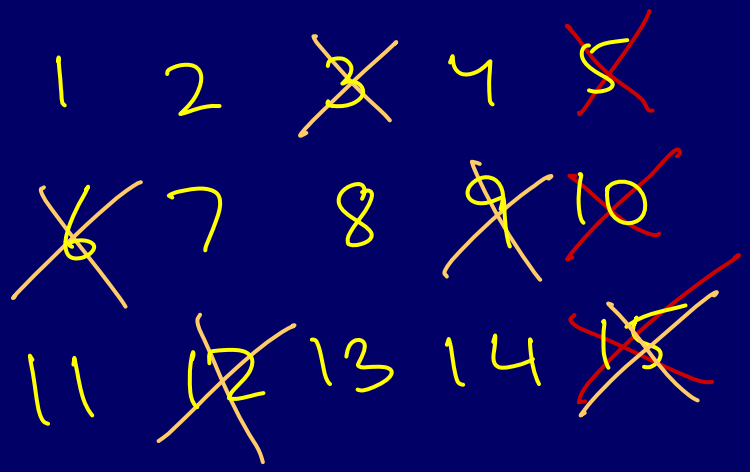
$*_{12}$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

Theorem: if p,q distinct primes then $\phi(pq) = (p-1)(q-1)$

$\phi(75) = 6 \cdot 4 = 24$

How about p = 3, q = 5?

$\phi(pq) = |\{x : 1 \le x \le pq-1 \text{ s.t. } \gcd(x, pq) = 1\}|$



$15 - 3 - 5 + 1 = 8 = 2 \cdot 4 = (3-1)(5-1)$

Theorem: if p,q distinct primes then $\phi(pq) = (p-1)(q-1)$

pq = # of numbers from 1 to pq

p = # of multiples of q up to pq

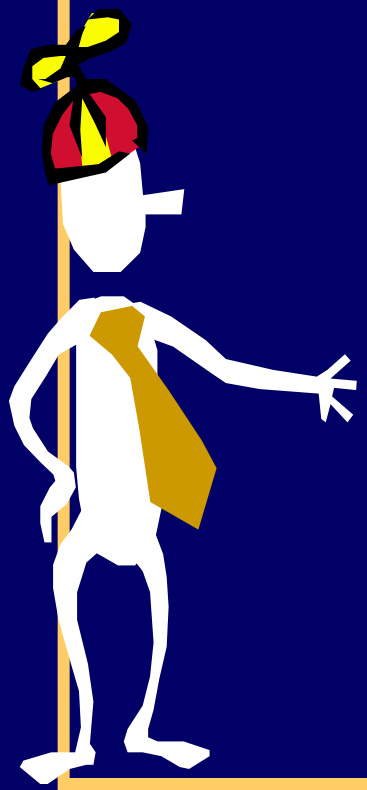
q = # of multiples of p up to pq

1 = # of multiple of <u>both</u> p and q up to pq

$$\phi(pq) = pq - p - q + 1 = (p-1)(q-1)$$

What if p, q, r are 3 distinct primes?
What is $\phi(p^2)$? if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$

Additive and Multicative Inverses



The additive inverse of $a \in Z_n$ is the unique $b \in Z_n$ such that $a +_n b \equiv_n 0$.

We denote this inverse by "-a".

It is trivial to calculate:
"-a" = (n-a). $(\mathrm{mod}\ n)$

The multiplicative inverse of $a \in Z_n^*$ is the unique $b \in Z_n^*$ such that

$$a *_n b \equiv_n 1.$$

We denote this inverse by "a^{-1}" or "$1/a$".

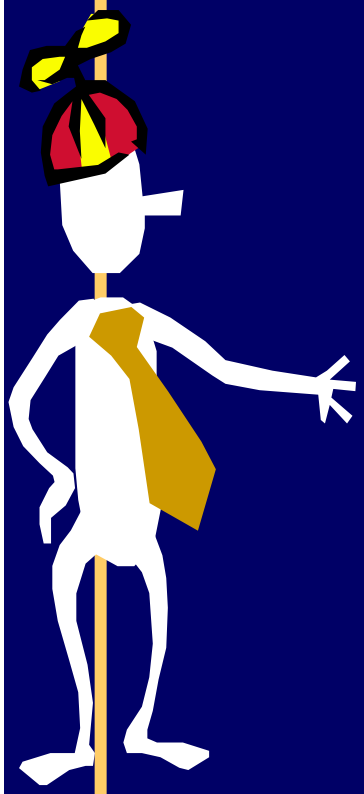
The unique inverse of "a" must exist because the "a" row contains a permutation of the elements and hence contains a unique 1.

*	1	b	3	4
1	1	2	3	4
2	2	4	1	3
a	3	1	4	2
4	4	3	2	1

$(\mathrm{mod}\ 5)$ $\quad 2^{-1} = 3$

Efficient algorithm to compute a^{-1} from a and n.

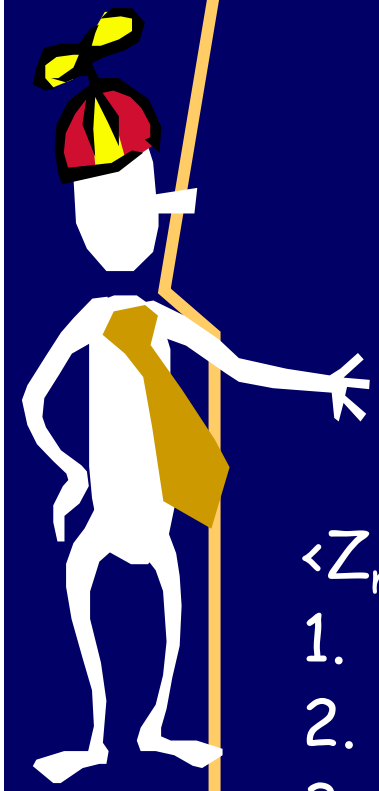
Run Extended Euclidean Algorithm on the numbers a and n.



It will give two integers r and s such that $ra + sn = \gcd(a,n) = 1$

Taking both sides modulo n, we obtain: $ra \equiv_n 1$

Output r, which is the inverse of a



$$Z_n = \{0, 1, 2, \ldots, n-1\}$$
$$Z_n^* = \{x \in Z_n \mid GCD(x,n) = 1\}$$

Define $+_n$ and $*_n$:

$a +_n b = (a+b \bmod n)$ $\qquad$ $a *_n b = (a*b \bmod n)$

$$c *_n (a +_n b) \equiv_n (c *_n a) +_n (c *_n b)$$

$\langle Z_n, +_n \rangle$

1. Closed
2. Associative
3. 0 is identity
4. Additive Inverses
5. Cancellation
6. Commutative

$\langle Z_n^*, *_n \rangle$

1. Closed
2. Associative
3. 1 is identity
4. Multiplicative Inverses
5. Cancellation
6. Commutative

Fundamental Lemmas until now

For x, y, a, b in Z_n, ($x \equiv_n y$) and ($a \equiv_n b$).
Then

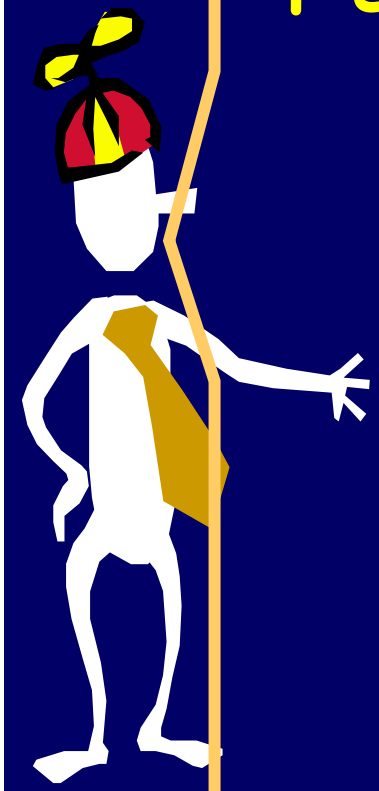
1) $x + a \equiv_n y + b$

2) $x - a \equiv_n y - b$

3) $x * a \equiv_n y * b$

For a,b,c in Z_n^*
then $ca \equiv_n cb \Rightarrow a \equiv_n b$

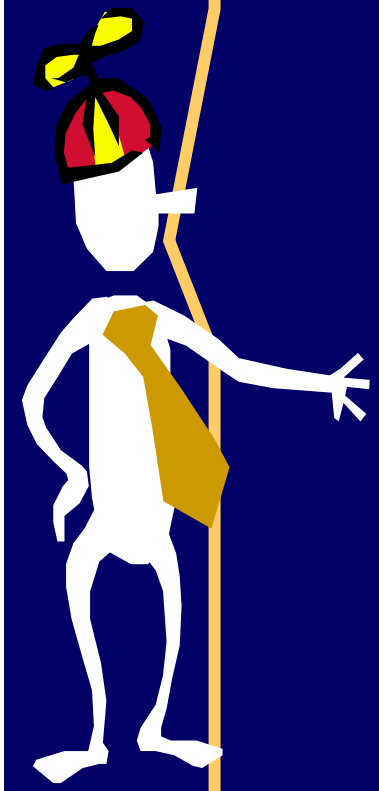
Fundamental lemma of powers?



If $(a \equiv_n b)$
Then $x^a \equiv_n x^b$?

NO!

$(2 \equiv_3 5)$, but it is not the case that: $2^2 \equiv_3 2^5$



By the permutation property, two names for the same set:

$$Z_n^* = aZ_n^*$$

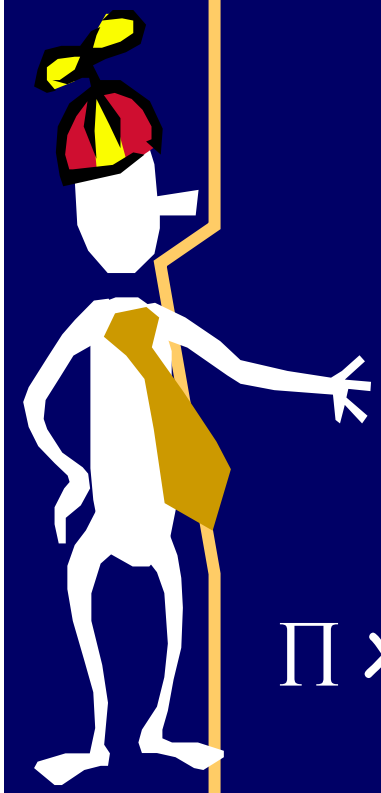
$Z_5^* = \{1, 2, 3, 4\}$ $\quad 3 \cdot Z_5^* = \{3, 1, 4, 2\}$

where

$$aZ_n^* = \{a *_n x \mid x \in Z_n^*\},\ a \in Z_n^*$$

Example: Z_5^*

*	1	2	3	4
1	1	2	3	4
2	2	4	1	3
a	3	1	4	2
4	4	3	2	1



Two products on the same set:

$$Z_n^* = aZ_n^*$$

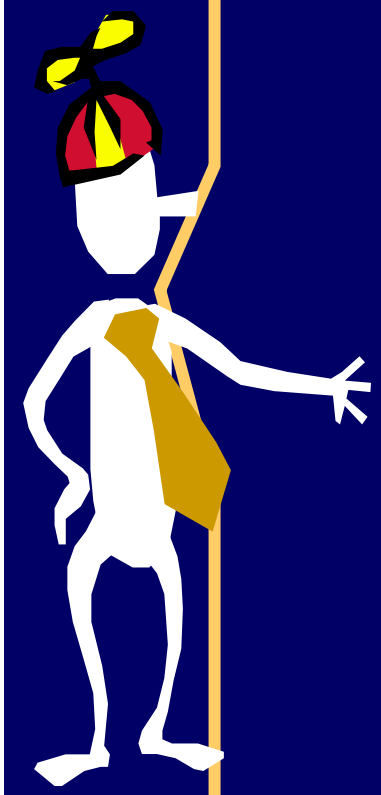
$$aZ_n^* = \{a *_n x \mid x \in Z_n^*\},\ a \in Z_n^*$$

$\prod x \equiv_n \prod ax$ [as x ranges over Z_n^*]

$\prod x \equiv_n \prod x\ (a^{\text{size of } Zn^*})$ [Commutativity]

$1 = a^{\text{size of } Zn^*}$ [Cancellation]

$$a^{\Phi(n)} = 1 \pmod{n}$$



Euler's Theorem

$$a \in Z_n^*,\ a^{\Phi(n)} \equiv_n 1$$

Fermat's Little Theorem

$$p \text{ prime}, a \in Z_p^* \Rightarrow a^{p-1} \equiv_p 1$$

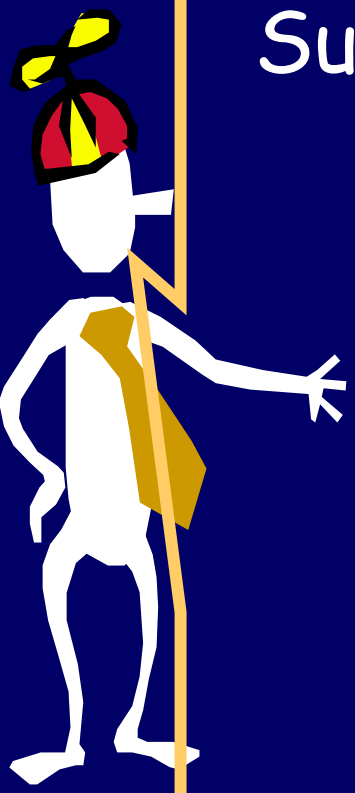
Fundamental lemma of powers.

Suppose $x \in Z_n^*$, and a,b,n are naturals.

If $a \equiv_{\Phi(n)} b$ Then $x^a \equiv_n x^b$

Equivalently,

$x^{a \bmod \Phi(n)} \equiv_n x^{b \bmod \Phi(n)}$



How do you calculate

$2^{4444444441} \bmod 5$

$2 \cdot 2^{444\cdots40} \pmod 5$

$2^{4444\cdots45} \bmod 5 = 2 \cdot 2^{44\cdots4} \pmod 5$

$(4\cdots4 \bmod \phi(5))$

$= 2 \cdot 2^{0} \pmod 5 = 2$

Fundamental lemma of powers.

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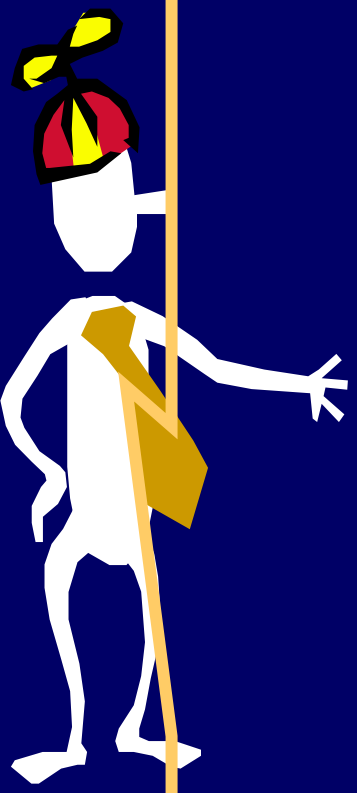
$x^a \pmod n = x^{a \bmod \Phi(n)} \pmod n$



Defining negative powers

Suppose $x \in Z_n^*$, and a,n are naturals.

x^{-a} is defined to be the multiplicative inverse of x^a

$(x^{-1})^a = x^{-a} = (x^a)^{-1}$

Rule of integer exponents

Suppose $x,y \in Z_n^*$, and a,b are integers.

$$(xy)^{-1} \equiv_n x^{-1} y^{-1}$$

$$X^a X^b \equiv_n X^{a+b}$$

$$Z_n = \{0, 1, 2, \ldots, n-1\}$$
$$Z_n^* = \{x \in Z_n \mid GCD(x,n) = 1\}$$

Quick raising to power.

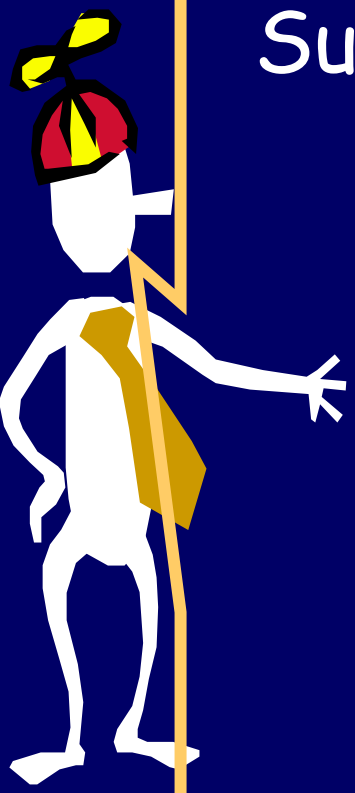
$\langle Z_n, +_n \rangle$

1. Closed
2. Associative
3. 0 is identity
4. Additive Inverses
   Fast + and -
5. Cancellation
6. Commutative

$\langle Z_n^*, *_n \rangle$

1. Closed
2. Associative
3. 1 is identity
4. Multiplicative Inverses
   Fast * and /
5. Cancellation
6. Commutative

Fundamental lemma of powers.



Suppose $x \in Z_n^*$, and a,b,n are naturals.

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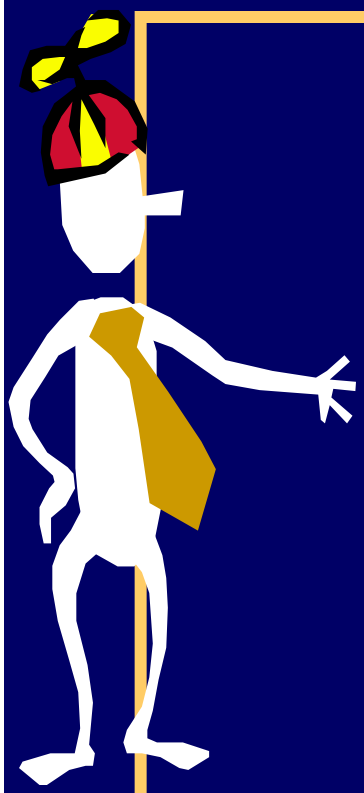


Euler Phi Function

$\Phi(n)$ = size of Z_n^*

p prime $\Rightarrow Z_p^* = \{1,2,3,\ldots,p-1\}$
$\Rightarrow \phi(p) = p-1$

$\phi(pq) = (p-1)(q-1)$
if p,q distinct primes

The RSA Cryptosystem

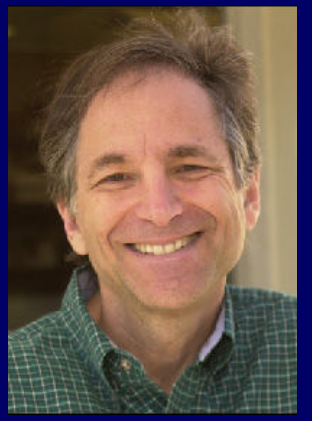
Rivest, Shamir, and Adelman (1978)

RSA is one of the most used cryptographic protocols on the net. Your browser uses it to establish a secure session with a site.

Back to our dramatis personae

Rivest

Shamir



Adleman

Euler

Fermat

The RSA Cryptosystem

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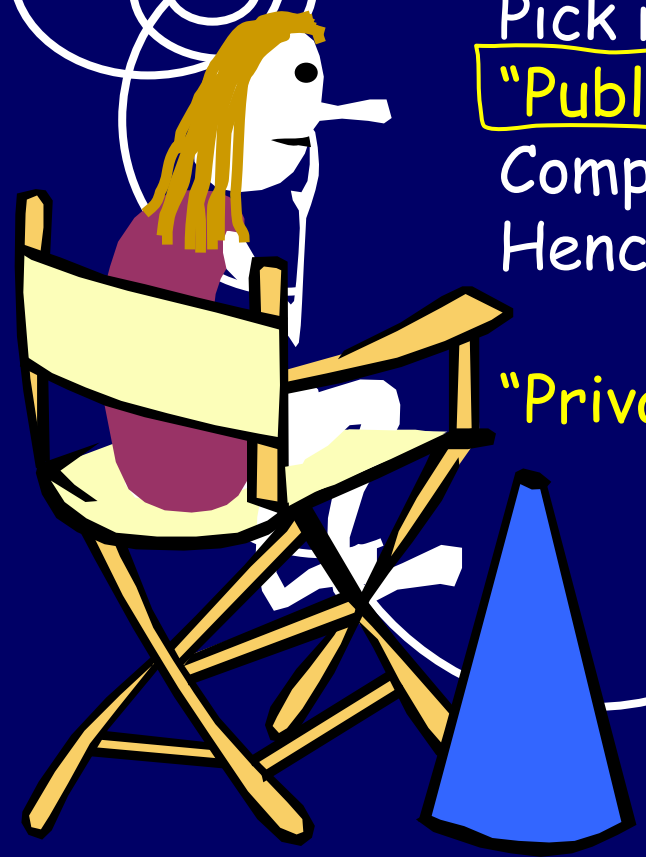


Pick secret, random large primes: p,q

"Publish": n = p*q

$= \phi(pq)$

$\phi(n) = \phi(p)\,\phi(q) = (p-1)*(q-1)$

Pick random $e \in Z^*_{\phi(n)}$

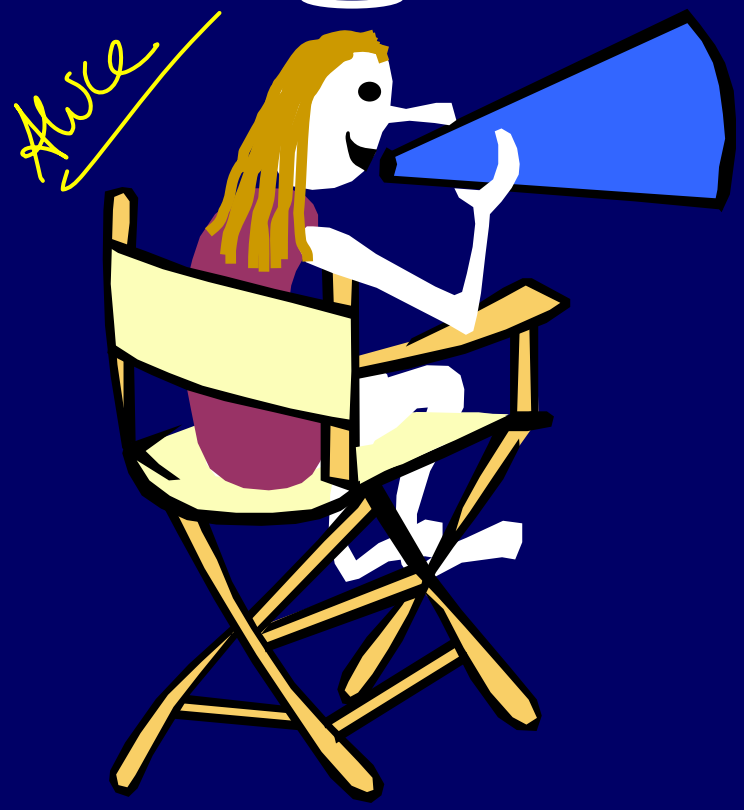
"Publish": e

Compute d = inverse of e in $Z^*_{\phi(n)}$

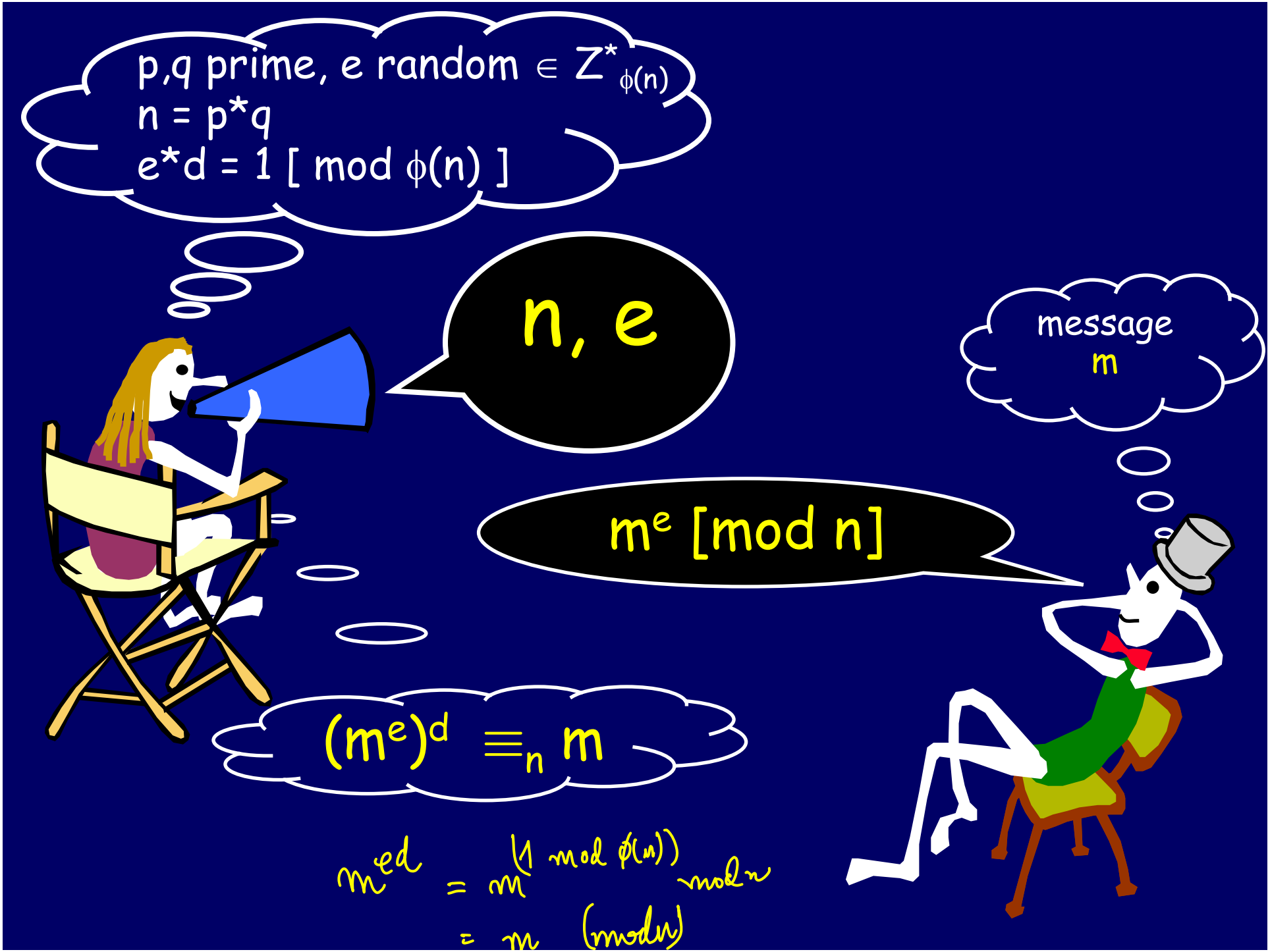
Hence, $e*d = 1 \ [\text{ mod } \phi(n)\ ]$

"Private Key": d

p,q random primes, e random $\in Z^*_{\phi(n)}$
$n = p*q$
$e*d = 1 \ [\ \text{mod}\ \phi(n)\ ]$



n,e is my public key. Use it to send me a message.


p,q prime, e random ∈ $Z^*_{\phi(n)}$
n = p*q
e*d = 1 [mod ϕ(n)]
n, e
message
m
m^e [mod n]
$(m^e)^d \equiv_n m$
$m^{ed} = m^{(1 \bmod \phi(n))} \bmod n$
$= m \pmod n$